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Cyclic algebras over p-adic curves

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Abstract

In this paper we study division algebras over the function fields of curves over $\mathbb{Q}_p$. The first and main tool is to view these fields as function fields over nonsingular S which are projective of relative dimension 1 over the p adic ring $\mathbb{Z}_p$. A previous paper showed such division algebras had index bounded by n^2 assuming the exponent was n and n was prime to p. In this paper we consider algebras of prime degree (and hence exponent) $q \neq p$ and show these algebras are cyclic. We also find a geometric criterion for a Brauer class to have index q.

Keywords: Division algebra; Cyclic algebra; Ramification

Introduction

In [S], this author studied division algebras over the following fields. Let K be a field finite over $\mathbb{Q}_p(t)$, for the p adic field $\mathbb{Q}_p$. That is, suppose there is a p adic field K' and a curve C defined over K' such that $K = K'(C)$. Let n be prime to p. In [S] we studied division algebras D/K (meaning K is the center of D) and showed that if their order in the Brauer group was n, then their degree was no more than n^2.

This paper is motivated by the idea that there are further interesting things to say about division algebras over these fields K. For example, suppose D/K has degree q^2, for a prime $q \neq p$, and order q in the Brauer group. The techniques of [S] show that, assuming K has a primitive q root of one, then D/K is an abelian crossed product (e.g., [LN], p. 37). For this and other more

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obvious reasons, it is of interest to study D/K of prime degree q. The important question is whether these D are cyclic algebras, and the answer we provide here is that such D are cyclic, whether or not there are q roots of one (5.1).

The first important step here, as in [S], is to observe that K is the function field of a regular surface S projective over $\mathrm{Spec}(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Thus much of this paper will have a geometric character, as the geometry imposed on S by D needs to be explicated and understood. Let me apologize in advance to geometers for the proofs I may provide for well known facts. Part of the intended audience of this paper consists of people primarily concerned with division algebras. I have chosen, therefore, to provide proofs of any facts that cannot be found in standard texts like Hartshorne and EGA.

Let me review briefly the structure of the paper. Our approach will be to prove as much as we can about Brauer classes over surfaces, and only use the strong condition on S above when needed at the end. In more detail, in this introductory section we review some material about Brauer groups, ramification, cyclic extensions, etc. Section 1 is a general geometry section. In the first half we review facts about surfaces S projective over $\mathbb{Z}_p$, and in the second half we consider a cohomology group $H^1(X, \mathcal{O}_P^*)$ over a much more general scheme X. The point is to do "divisor theory" while controlling behavior at finitely many points. In Section 2 we assume the ground field has a primitive q root of one, and study the geometry of the ramification of a Brauer class of order q. In Section 3 we remove the assumption on roots of unity. In Section 4 we consider the behavior of "residual" classes, and in Section 5 we prove the main results.

Let q be a fixed prime unequal to p throughout this paper. Let μ_q be the group of q roots of one over any field. We denote by G_F the absolute Galois group of a field F. That is, G_F is the Galois group of F in its separable closure. If $\mu_q \subset F^*$, there is a pairing $G_F \times F^* \to \mu_q$ defined by sending $(\sigma, u) \to \sigma(u^{1/q})/u^{1/q}$. If F is a finite field containing μ_q, then the Frobenius defines a canonical generator of G_F and so the Frobenius defines a homomorphism $Fr: F^* \to \mu_q$.

Recall that if K is a field, the Brauer group $\mathrm{Br}(K)$ consists of equivalence classes $[A]$ of central simple algebras A/K, and each such class contains a unique division algebra. If $\alpha \in \mathrm{Br}(K)$, then the order of α is its order in the Brauer group, and the index of α is the degree (i.e., square root of the dimension) of the associated division algebra over K. A cyclic algebra is a central simple algebra A/K of degree n containing L where L/K is cyclic Galois of degree n (L need not be a field). All cyclic algebras have the form $A = \Delta(L/K, \sigma, a)$ where L/K is cyclic Galois, $\sigma \in \mathrm{Gal}(L/K)$ is a generator, and $a \in K^*$ (e.g., [LN], p. 49). Note that $\Delta(L/K, \sigma, a) \cong \Delta(L/K, \sigma^s, a^s)$ where s is prime to the degree of L/K. If $K' \supset K$ is a field extension, recall that $\alpha \in \mathrm{Br}(K)$ is split by K' if it is in the kernel of the natural map $\mathrm{Br}(K) \to \mathrm{Br}(K')$ given by $[A] \to [A \otimes_K K']$. Perhaps the most important fact about cyclic algebras we need is the well known theorem of Albert:

Proposition 0.1. *Suppose A/K is a central simple algebra of prime degree q. Then A is a cyclic algebra if and only if there is a $\pi \in K^*$ such that $K' = K(\pi^{1/q})$ splits $[A]$.*

Proof. The description of cyclic algebras above shows that they contain such Kummer maximal subfields, and such a subfield necessarily splits A. Thus the "only if" part is done. If such a K' splits A, by [LN], p. 25, it is isomorphic to a subfield of A. This result now follows from [A], p. 77. □

When F contains ρ, a generator of μ_q, all cyclic algebras over F have the following form. If $a, b \in F^*$ then one can define the symbol algebra $(a, b)_{q,F,\rho}$ as the central simple F algebra generated by x, y satisfying the relations $x^q = a$, $y^q = b$ and $yx = \rho xy$. Just as with general cyclic algebras, we have that $(a,b)_{q,F,\rho} \cong (a,b^s)_{q,F,\rho^s}$ where s is prime to q. We will often drop all or a subset of the q, F, ρ subscript because q is fixed throughout the paper, F is usually clear, and ρ is often fixed in advance. We will also write $(a,b) \in \mathrm{Br}(F)$ for the Brauer group element represented by the algebra (a,b) (and called a *symbol class*).

If R is a discrete valuation domain with field of fractions $q(R) = K$ and residue field F of characteristic p, then there is the well known ramification map (e.g., [Se], p. 186)

$$\mathrm{ram}: \mathrm{Br}(K)' \to \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})'$$

where for a torsion abelian group A, A' refers to the prime p part of A. Note that any q order element $\phi \in \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})$ can be represented by a pair $L/F, \sigma$ where the kernel of ϕ has fixed field L and σ is the generator of $C_q = \mathrm{Gal}(L/F)$ which maps to $1/q + \mathbb{Z}$. In this paper ramification will be frequently written this way.

This ramification map is almost completely determined by the following two observations. First, let $\tilde{K}$ be the completion of K with respect to R. Then the ramification map factors as $\mathrm{Br}(K)' \to \mathrm{Br}(\tilde{K})' \to \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})'$ where the first map is the usual restriction on Brauer groups and the second map is the ramification associated to the valuation on $\tilde{K}$. Second, assume $K = \tilde{K}$ is complete. Suppose L/K is cyclic unramified of degree prime to p, with generator $\sigma \in \mathrm{Gal}(L/K)$. Let $\bar{L}/F, \bar{\sigma}$ be the residue extension and corresponding generator. Then the ramification of the cyclic algebra $\Delta(L/K, \sigma, \pi)$ is $\bar{L}/F, \bar{\sigma}$ when π is any prime element of K.

In particular, assume F contains μ_q and we fix a generator $\rho \in \mu_q$. Then the q torsion part of $\mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})$ can be identified with $F^*/(F^*)^q$. In detail, the pair $L/F, \sigma$ is identified with $a(F^*)^q$ where $L = F(a^{1/q})$ and $\sigma(a^{1/q})/a^{1/q} = \rho$. Thus a q torsion element of $\mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})$ will sometimes represented by an element of $F^*/(F^*)^q$ or by $a^{1/q}$ for some $a \in F^*$. With all of this, there is an easy way to write the ramification of a symbol class (a,b). The following result is well known and computable from the above or, for example, [LN], p. 68.

Lemma 0.2. *Suppose $R \subset K$, F are as above, $\rho \in \mu_q \subset K$ is fixed, and $(a,b) \in \mathrm{Br}(K)$ is a symbol class. Let $d: K^* \to \mathbb{Z}$ be the valuation associated to R. Then* $\mathrm{ram}((a,b)) = (\bar{u})^{1/q}$ *where $u = (-1)^{d(a)d(b)} a^{d(b)}/b^{d(a)}$ and where $\bar{u}$ refers to the image of u in F^*.*

Suppose we have a field K which is the function field of a normal integral scheme X of finite type over a Noetherian ring. Let $\alpha \in \mathrm{Br}(K)$. For each irreducible divisor $D \subset X$ let R_D be the stalk of the structure sheaf of X, which is a discrete valuation domain. There are only finitely many D_i where α has nontrivial ramification $L_i/F(D_i), \sigma_i$. The set of D_i where α is ramified is called the *ramification locus* of α. The set of D_i paired with the ramification $L_i/F(D_i), \sigma_i$ of α at each D_i is called the *ramification data* of α.

Much of this paper is about splitting ramification so it is important we describe how this is done. Let $R \subset K$ be a discrete valuation domain of K (meaning K is the field of fractions of R) and let F be the residue field of R. Let L/K be a finite separable extension field and let $\{S_i\}$ be the (necessarily finite) set of discrete valuation domains of L which extend R. Let F_i be the residue field of S_i and $e_i = e(S_i/R)$ the ramification index. Let $\mathrm{ram}_i: \mathrm{Br}(L)' \to \mathrm{Hom}(G_{F_i}, \mathbb{Q}/\mathbb{Z})'$ and $\mathrm{ram}: \mathrm{Br}(K)' \to \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})'$ be the respective ramification maps.

Lemma 0.3. *The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Br}(L)' & \xrightarrow{\sum \mathrm{ram}_i} & \oplus \mathrm{Hom}(G_{F_i}, \mathbb{Q}/\mathbb{Z})' \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\sum e_i} \\ \mathrm{Br}(K) & \xrightarrow{\mathrm{ram}} & \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})' \end{array}$$

where ι is the restriction and $e_i : \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})' \to \mathrm{Hom}(G_{F_i}, \mathbb{Q}/\mathbb{Z})'$ is the natural map multiplied by the integer e_i.

If $\iota(\alpha)$ is unramified at all S_i we say L/K *splits all the ramification of* α *at* R. We are particularly interested in the case L/K above is a field extension of prime degree q unequal to the residue characteristic.

Corollary 0.4. *Let L/K in Lemma* 0.3 *be of prime degree q unequal to the residue characteristic. Assume $\alpha \in \mathrm{Br}(K)$ has ramification $F'/F, \sigma$ of order q. Then $\iota(\alpha)$ is unramified at all the S_i if and only if there is a unique extension, S, of R to L and one of the following two exclusive conditions hold*:

(i) *L/K is totally and tamely ramified.*
(ii) *L/K is unramified at R and the residue field of S is F'.*

Proof. If S is not unique, all ramification degrees and all residue extension degrees are prime to q and L cannot split the ramification at any extension. On the other hand, suppose R extends to a unique S. If L/K is ramified, $q = e(S/R)$, and $\iota(\alpha)$ has zero ramification at S. If S/R is unramified, let F'' be the residue field of S, so F''/F is of degree q. Then $F'' \supset F'$ if and only if $F'' = F'$ if and only if $\iota(\alpha)$ has zero ramification at S. □

If L/K as in Corollary 0.4 satisfies (i) we say it splits α *by ramification* and if L/K satisfies (ii) we say it splits α *by residues*.

Let us make one more definition. Suppose $\alpha \in \mathrm{Br}(K)$, K has a discrete valuation R, and L/K splits the ramification of α at R and is totally ramified, which includes that R extends uniquely. If S is that unique extension, and $\alpha_L = \alpha \otimes_K L$ is the image of α in $\mathrm{Br}(L)$, then Lemma 0.3 shows that $\alpha_L \in \mathrm{Br}(S)$. If F is the residue field of S and hence of R, then α_L has an image $\beta_R \in \mathrm{Br}(F)$ we call the *residual Brauer class* of α at R with respect to L.

We can make the following observation about β_R.

Proposition 0.5. *Suppose α, R, K and L are as above, and let $F'/F, \sigma$ be the nonzero ramification of α at R. Assume L/K has degree q. Suppose α has index q, meaning it is represented by a division algebra of degree q. Then the residual Brauer class β_R, with respect to any L, is split by F'.*

Proof. At the completion α must still have index q, so it suffices to prove this under the assumption that K is complete with respect to R. In addition, it suffices to show this after we adjoin a q root of one. Thus we may assume K contains a primitive q root of one. Since L/K is totally and tamely ramified, it is cyclic of degree q. But then $\alpha = \alpha' + (K'/K, \pi)_{q,K}$ where K'/K is the unramified extension with residue extension F'/F, $L = K(\pi^{1/q})$ and $\alpha' \in \mathrm{Br}(R)$ with image β_R. By, e.g., [JW], p. 161, if F' does not split β_R then α has index bigger than q. $\square$

In the rest of this paper, the R of Proposition 0.5 will sometimes be defined by a curve C on a surface S, and in that case we will write the residual Brauer class as β_C.

It will later be important to determine how this residual class β_R depends on the choice of L. To this end, let $R \subset K$ be a discrete valuation domain with field of fractions K as above.

Proposition 0.6. *Suppose $\alpha \in \mathrm{Br}(K)$ of order q has ramification $F'/F, \sigma$ at R. Let $L = K(\pi^{1/q})$ for π a prime of R. Also set $L' = K((u\pi)^{1/q})$ where u is a unit of R. Let β_R, β'_R be the respective residual classes of α defined by L and L'. Then $\beta'_R = \beta_R + \Delta(F'/F, \sigma, \bar{u}^{-1})$ where $\bar{u}$ is the image of u in F^*.*

Proof. Just as above, to prove this we can assume K is complete with respect to R. Let K'/K be unramified with residue extension F'/F. Then $\alpha = \alpha' + \Delta(K'/K, \sigma, \pi) = \alpha' + \Delta(K'/K, \sigma, u^{-1}) + \Delta(K'/K, \sigma, u\pi)$. Since β_R is the image of α' and β'_R is the image of $\alpha' + \Delta(K'/K, \sigma, u^{-1})$, we are done. $\square$

Sometimes it will not be convenient to have L written as $K(\pi^{1/q})$ with π a prime but only with π having prime to q valuation. The following is obvious.

Corollary 0.7. *Let $\alpha \in \mathrm{Br}(K)$ and R, F, $F'/F, \sigma$ be as above. Suppose v is the valuation of R and $\pi \in K$ satisfies $v(\pi) = s$ which is prime to q. Set $L = K(\pi^{1/q})$ and let β_R be the corresponding residual Brauer class. Suppose $u \in R^*$ has image and $L' = K((u\pi)^{1/q})$. If β'_R is the residual class with respect to L', then $\beta'_R = \beta_R + \Delta(F'/F, \sigma, \bar{u}^{-t})$ where $st - 1$ is divisible by q.*

Remark. Suppose $\alpha \in \mathrm{Br}(K)$, R, and F'/F are as in Corollary 0.7. If α has index q, we know by Proposition 0.5 that F' splits β_R. The converse is false but Corollary 0.7 makes the following clear. If for one choice of L, β_R is split by F', then this is true for all choices of L. When this happens, we say the *residual classes of α are split by the ramification*.

Suppose next that $K = k(C)$ is the function field of a curve over a finite field k. Then the set of discrete valuations on K is exactly the set of points on C. If R is any such discrete valuation, then the residue field R/P is a finite field and hence $\mathrm{Hom}(G_{R/P}, \mathbb{Q}/\mathbb{Z})$ can be identified with $\mathbb{Q}/\mathbb{Z}$ using evaluation on the Frobenius. Thus there is a map $\mathrm{Br}(K) \to \bigoplus_{P \in C} \mathbb{Q}/\mathbb{Z}$. Note that since all finite fields are perfect, we can define this map even on the p primary part of $\mathrm{Br}(K)$ (e.g., [Se], p. 186). From class field theory we know (e.g., [R], p. 277):

Theorem 0.8. *There is an exact sequence*

$$0 \to \mathrm{Br}(K) \to \bigoplus_{P \in C} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

where $\bigoplus_{P \in C} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the summation map.

If $\alpha \in \mathrm{Br}(K)$ then the image of α in the copy of $\mathbb{Q}/\mathbb{Z}$ corresponding to $P \in C$ we call the *residue* of α at P.

One consequence of Theorem 0.8 is that, over $K = k(C)$ as in Theorem 0.8, splitting all ramification is equivalent to splitting. This is false in general, though of course splitting implies splitting all ramification. The most important fact about the fields K that are the focus of this paper is that for them also, splitting all ramification implies splitting.

Theorem 0.9. *Suppose S is a surface projective and regular over* $\mathrm{Spec}(\mathbb{Z}_p)$. *Let K be the function field of S. If $\alpha \in \mathrm{Br}(K)$ has trivial ramification at all discrete valuations lying over $\mathbb{Z}_p$, then $\alpha = 0$.*

Proof. It suffices to show $\mathrm{Br}(S) = 0$. By [G], p. 98, $\mathrm{Br}(S) \cong \mathrm{Br}(\bar{S})$ and so it suffices to show $\mathrm{Br}(\bar{S}) = 0$. By, for example, the argument of [S], p. 40, it suffices to show $\mathrm{Br}(C) = 0$ for C any complete nonsingular curve over a finite field, and this is a part of Theorem 0.8. $\square$

Having discussed ramification of algebras, let us consider that of cyclic extensions. Let R be a discrete valuation domain with residue field $F = R/\mathcal{M}$ and field of fractions $K = q(R)$. Suppose L/K is a cyclic Galois extension of prime degree q with generator σ of its Galois group. We assume q is not the characteristic of F and μ_q is the group of q roots of one over K. We need to define the ramification $\rho \in \mu_q$ of $L/K, \sigma$ at R. If L/K is unramified, of course $\rho = 1$. If L/K is ramified, let $\tilde{K}$ be the completion of K with respect to R and $\tilde{L} = L \otimes_K \tilde{K}$. Then $\tilde{L}$ is a field. Since $\tilde{L}/\tilde{K}$ is a totally and tamely ramified extension, it follows that $\mu_q \subset \tilde{K}$ and hence $\mu_q \subset F$. Furthermore, $\tilde{L}/\tilde{K}, \sigma$ has the form $\tilde{K}((\pi)^{1/q})$ for some prime element π. Note that $\pi^{1/q} \in \tilde{L}$ is a prime element of $\tilde{L}$. We set the ramification $\rho = \sigma(\pi^{1/q})/\pi^{1/q}$ viewed as a root of unity over F. It is useful to note that this ρ can be defined using any prime element of $\tilde{L}$ and hence of L. In fact, suppose δ is a prime element of $\tilde{L}$. Then $\delta = u\pi^{1/q}$ for a unit u of $\tilde{L}$. Since σ acts trivially on the residue field of $\tilde{L}$, it follows that ρ is the image of $\sigma(\delta)/\delta$ in the residue field of L.

The ramification of a cyclic extension can be used to express the ramification of a cyclic algebra as follows. Suppose K is a field with a discrete valuation domain R and $\alpha = \Delta(L/K, \sigma, u)$ is of degree q where $u \in R^*$ and u has image $\bar{u}$ in the residue field F of R. If L/K is unramified then α has 0 ramification. If not, F contains a primitive q root of one. Let ρ be the ramification of $L/K, \sigma$ at R. The following is easy.

Lemma 0.10. *The ramification of α is described by $F(\bar{u}^{1/q}), \sigma'$ where $\sigma'(\bar{u}^{-1/q})/\bar{u}^{-1/q}) = \rho$, and ρ is the ramification of $L/K, \sigma$ at R.*

In a couple of places in this paper we will need to know certain discrete valuations exist, beyond those that arise from blowing up points. To this end, let R be a local domain with field of fractions $K = q(R)$. A discrete valuation $d : K^* \to \mathbb{Z}$ of K is said to *lie over R* if $d(R) \geqslant 0$ and $d(R) \neq \{0\}$. If $P = \{r \in R \mid d(r) > 0\}$ then P is a nonzero prime and we say d lies over P. If R is a domain and L/K splits all the ramification at any discrete valuation lying over R we say L/K splits all the ramification of α at R.

Lemma 0.11. *Suppose R is a two-dimensional local regular domain with parameters π, δ, residue field $F = R/M$, and field of fractions K. Let T be transcendental over K. Suppose $a, b \in \mathbb{Z}$ are positive integers. Then there is a valuation $d : K(T)^* \to \mathbb{Z}$ on $K(T)$ with the following properties. First of all, $d(T) = 1$, $d(\pi) = a$ and $d(\delta) = b$. Secondly, the residue field of d is*

$F(\pi', \delta')$ where π' is the image of π/T^a, δ' is the image of δ/T^b, and π', δ' are transcendental over F.

Proof. Form the polynomial ring $R[T, \pi'', \delta'']$. Let R' be the localization of this polynomial ring at the maximal ideal generated by $\pi, \delta, T, \pi'', \delta''$, so R' is also a regular local domain. Then $T^a\pi'' - \pi, T^b\delta'' - \delta, T, \pi'', \delta''$ clearly generate the maximal ideal of R' and hence form an R sequence. Let $R_1 = R'/(T^a\pi'' - \pi, T^b\delta'' - \delta)$ which is a regular local ring with parameters we can identify with T, π'', δ''. Then $R \subset R_1$ and R_1 has field of fractions $K(T)$. Let S be the discrete valuation ring formed by localizing R_1 at its prime T and let d be the associated valuation. Clearly $d(\pi) = a$, $d(\delta) = b$, and the residue field of S is $k(\pi', \delta')$ where π', δ' are the images of π'', δ'' and are transcendental over F. □

We will make frequent use of the well-known fact (e.g., [E], p. 487) that a regular local ring is a UFD. In fact, we will need a very slight generalization:

Lemma 0.12. *Suppose R is a regular semilocal ring. Then R is a UFD.*

Proof. It suffices to show that every height one prime P is principal. But if $M \subset R$ is a maximal ideal, PR_M is a height one prime and hence principal. That is, P is locally free, therefore projective, and therefore free of rank one since R is semilocal. □

1. The surface

Let $S \to \mathrm{Spec}(\mathbb{Z}_p)$ be projective, regular, excellent, flat of relative dimension one. Let $\bar{S}$ be the set theoretic inverse image of the closed point of $\mathrm{Spec}(\mathbb{Z}_p)$ with the reduced induced structure. We also assume $\bar{S}$ has nonsingular components and only normal crossings. In this section we review some general facts about this situation, which we will apply to the Brauer group in subsequent sections.

First of all let us consider closed points on S, by which we mean codimension 2 closed points. It is easy to see that all such points lie on $\bar{S}$. Next, we consider codimension 1 points which we call curves. $\bar{S}$ is the finite union of curves. If $E \subset S$ is any other curve, it lies over the generic point of $\mathbb{Z}_p$ and thus defines a point of the $\mathbb{Q}_p$ curve $S \times_{\mathbb{Z}_p} \mathbb{Q}_p$. The restriction $E \to \mathrm{Spec}\,\mathbb{Z}_p$ is surjective, projective, of relative dimension 0 and so must be finite. Thus ([H], p. 280) E is affine with affine ring, R, a domain finite over $\mathbb{Z}_p$. The Henselian property of $\mathbb{Z}_p$ shows that R has 0 and one other prime ideal which lies over $p\mathbb{Z}_p$. That is, E has a generic point and exactly one closed point. We call such E geometric curves of S.

We observe and recall the well-known fact that points of $\bar{S}$ lift nicely to S.

Lemma 1.1.

(a) *Let $P \in \bar{S}$ be a (nonsingular) point on a single component. There is a nonsingular geometric curve $E \subset S$ such that P is the multiplicity one intersection of E and $\bar{S}$.*
(b) *If $P \in \bar{S}$ is a point on two components, there is a nonsingular geometric E which meets each component with multiplicity one at P.*

Proof. Let $R = \mathcal{O}_{S,P}$ be the stalk at P and M_P the maximal ideal. In (a), $p = \delta^r u$ where $u \in R^*$ and δ is a regular prime of R. There is an $x \in R$ such that $(\delta, x) = M_P$. Then $R/(x)$ is a DVR, contains $\mathbb{Z}_p$, and so must be the integral closure of $\mathbb{Z}_p$ in the field of fractions of $R/(x)$. In particular, $R/(x)$ is finite over $\mathbb{Z}_p$. If $\mathrm{Spec}(R') \subset S$ is an affine open set containing P, then $R' \subset R$ and R is a localization of R'. $R'/((x) \cap R') \subset R/(x)$ is also finite over $\mathbb{Z}_p$ and so has a unique maximal ideal. The extension $R'/((x) \cap R') \subset R/(x)$ is localization at that maximal ideal and so $R'/((x) \cap R') = R/(x)$ and this ring represents a nonsingular curve geometric curve E in S with multiplicity one intersection with $\bar{S}$ at P. Since E has a single closed point, this is the only place it intersects $\bar{S}$.

In (b), $p = \delta^r \delta'^s u$ where $u \in R^*$ and (δ, δ') is the maximal ideal of R. We can now choose $x = \delta + \delta'$ and proceed as above. □

The next issue to concern us is the relation of $\mathrm{Pic}(S)$ and $\mathrm{Pic}(\bar{S})$. There is a natural map $\mathrm{Pic}(S) \to \mathrm{Pic}(\bar{S})$ which cannot be an isomorphism but is close enough for our needs.

Theorem 1.2. *Suppose $\pi : S \to \mathrm{Spec}(\mathbb{Z}_p)$ and $\iota : \bar{S} \to S$ are as above. Let m be an integer prime to p. Then the induced map $\mathrm{Pic}(S) \to \mathrm{Pic}(\bar{S})$ is a surjection and induces an isomorphism $\mathrm{Pic}(S)/m\,\mathrm{Pic}(S) \cong \mathrm{Pic}(\bar{S})/m\,\mathrm{Pic}(\bar{S})$.*

We begin the proof with a proposition.

Proposition 1.3.

(i) *Let X be a scheme and $\mathcal{J} \subset \mathcal{O}_X$ an ideal sheaf. Let $f : Y \to X$ be the closed subscheme defined by $\mathcal{J}$. Suppose $\mathcal{F}$ is a coherent sheaf on X with $\mathcal{J}\mathcal{F} = 0$. Then $H^i(X, \mathcal{F}) = H^i(Y, \mathcal{F})$. It is also true that $H^i(X, (\mathcal{O}_X/\mathcal{J})^*) = H^i(Y, \mathcal{O}_Y^*)$.*

(ii) *Let X be a scheme and $\mathcal{J} \subset \mathcal{O}_X$ a nilpotent ideal sheaf. Let $f : Y \to X$ be the closed subscheme defined by $\mathcal{J}$. Assume Y has dimension one, and that the integer m is invertible in $\mathcal{O}_X$. Then $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is surjective and induces an isomorphism $\mathrm{Pic}(X)/m\,\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)/m\,\mathrm{Pic}(Y)$.*

Proof. To prove (i), note that $f_* f^*(\mathcal{F}) = \mathcal{F}$ and f is affine so Exer. 8.2, p. 252, of [H] shows this. The last sentence of (i) follows similarly.

Turning to (ii), by induction we may assume $\mathcal{J}^2 = 0$. There is an exact sequence of abelian group sheaves on X:

$$1 \to \mathcal{J} \to \mathcal{O}_X^* \to (\mathcal{O}_X/J)^* \to 1.$$

By (i) and [H], p. 208, $H^2(Y, \mathcal{J}) = 0$ and $\mathrm{Pic}(Y) = H^1(X, (\mathcal{O}_X/\mathcal{J})^*)$. It follows from the long exact sequence and (i) that $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is surjective. Also $H^1(X, \mathcal{J})$ is a module over the ring of global sections of X, implying that multiplication by m is an isomorphism. If $\alpha \in \mathrm{Pic}(X)$ maps to $m\,\mathrm{Pic}(Y)$, then by the surjectivity, there is a α' such that $\alpha - m\alpha'$ is the image of $\beta \in H^1(X, \mathcal{J})$. Since $\beta = m\beta'$ for a unique β', we have $\alpha = m\alpha' + m\beta''$ where β'' is the image of β'. □

We now turn to the proof of Theorem 1.2.

Proof. By Proposition 1.3, we can replace $\bar{S}$ with $S_1 \subset S$, the subscheme defined by $p\mathcal{O}_S$. Let S_n be the subscheme defined by $p^n\mathcal{O}_S$. Let $\mathcal{I}_n \in \mathrm{Pic}(S_n)$ be a previously defined line bundle. By Proposition 1.3, there is a line bundle $\mathcal{I}_{n+1}$ on S_{n+1} such that $\mathcal{I}_{n+1}/p^n\mathcal{I}_{n+1} = \mathcal{I}_n$.

By the Grothendieck existence theorem ([EGA], III 5.1.6) there is a line bundle $\mathcal{J}$ on S with $\mathcal{J}/p^n\mathcal{J} = \mathcal{I}_n$.

There is another way to view this surjectivity result. Since $\bar{S}$ is a union of smooth curves with normal crossings, an element of $\mathrm{Pic}(\bar{S})$ can be represented as a Cartier Divisor and hence as a sum of points on these curves that avoid the intersection points (use Proposition 1.5 without circularity). Let P be one of these points. Choose E as in Lemma 1.1. Then E defines a divisor, and hence an element of $\mathrm{Pic}(S)$ which is the preimage of the element of $\mathrm{Pic}(\bar{S})$ corresponding to P.

Next, we turn to the injectivity modulo m powers. Suppose $\mathcal{J} \in \mathrm{Pic}(S)$ maps to $\mathcal{I}^m \in \mathrm{Pic}(\bar{S})$. Then by lifting $\mathcal{I}$ we may assume $\mathcal{J}$ maps to the identity in $\mathrm{Pic}(\bar{S})$. That is, it suffices to show that the kernel of $\mathrm{Pic}(S) \to \mathrm{Pic}(\bar{S})$ is m divisible. By the above, $\mathcal{J}/p^n\mathcal{J} \cong (\mathcal{I}_n)^m$ for a unique line bundle $\mathcal{I}_n$ and so the existence theorem applied to the $\mathcal{I}_n$ show that there is an $\mathcal{I}$ with $\mathcal{J} \cong \mathcal{I}^m$.

Alternatively, the Kummer exact sequence shows that $\mathrm{Pic}(S)/m\,\mathrm{Pic}(S) \cong H^2_{et}(S, \mu_m)$ and $\mathrm{Pic}(\bar{S})/m\,\mathrm{Pic}(\bar{S}) \cong H^2_{et}(\bar{S}, \mu_m)$ (here we use that $H^2_{et}(S, \mathcal{O}^*) = 0 = H^2_{et}(\bar{S}, \mathcal{O}^*)$) The result follows from proper base change (e.g., [Mi], p. 223). □

We need a variation of Theorem 1.2 where we have some control over values of functions at finitely many points. To this end, let X be a scheme of finite type over a Noetherian ring A, and $P_1, \ldots, P_r$ a finite set of closed points each of which we write as $\iota_l : k(P_l) \to X$. In our application X will be either S or $\bar{S}$ so we will assume X is projective over a Noetherian domain and is reduced. Form the sheaf $\mathcal{P}^* = \bigoplus_l \iota_l^* k(P_l)^*$. There is a surjective morphism of sheaves $\mathcal{O}^* = \mathcal{O}^*_X \to \mathcal{P}^*$ which is just evaluation and we let $\mathcal{O}^*_P$ be the kernel. Let $\mathcal{K}$ be the sheaf of total quotient rings of X and $\mathcal{K}^*$ the group of units of $\mathcal{K}$. There are embeddings $\mathcal{O}^*_P \subset \mathcal{O}^* \subset \mathcal{K}^*$. Thus we have a exact sequence of sheaves

$$0 \to \mathcal{P}^* \to \mathcal{K}^*/\mathcal{O}^*_P \to \mathcal{K}^*/\mathcal{O}^* \to 0.$$

Since $\mathcal{P}^*$ and $\mathcal{K}^*$ are flasque we know $H^1(X, \mathcal{P}^*) = 0 = H^1(X, \mathcal{K}^*)$. Clearly $H^0(X, \mathcal{P}^*) = \bigoplus_l k(P_l)^*$. We set $K^* = H^0(X, \mathcal{K}^*)$. There are natural maps $K^* \to H^0(X, \mathcal{K}^*/\mathcal{O}^*_P)$ and $\bigoplus_l k(P_l)^* \to H^0(X, \mathcal{K}^*/\mathcal{O}^*_P)$, the later of which is an injection. The intersection, in $H^0(X, \mathcal{K}/\mathcal{O}^*_P)$, of the images of these maps we call k^* and we can identify k^* with the corresponding subgroup of $\bigoplus_l k(P_l)^*$. We have the exact diagram:

$$\begin{tikzcd}
 & 0 & 0 & \\
K^* \arrow[r] & H^0(X, \mathcal{K}^*/\mathcal{O}^*) \arrow[r] \arrow[u] & H^1(X, \mathcal{O}^*) \arrow[r] \arrow[u] & 0 \\
K^* \arrow[r] \arrow[u, equal] & H^0(X, \mathcal{K}^*/\mathcal{O}^*_P) \arrow[r] \arrow[u] & H^1(X, \mathcal{O}^*_P) \arrow[r] \arrow[u] & 0 \\
 & H^0(X, \mathcal{P}^*) \arrow[r] \arrow[u] & H^0(X, \mathcal{P}^*)/k^* \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Our goal is to interpret, a bit, $H^0(X, \mathcal{K}^*/\mathcal{O}_P^*)$ and $H^1(X, \mathcal{O}_P^*)$. Of course the former consists of equivalence classes of sets of pairs $\{(U_j, f_j)\}$ where $f_i \in \mathcal{K}^*(U_i)$, on $U_i \cap U_j$ the ratio f_i/f_j is a unit, and this unit maps to 1 at all $P_l \in U_i \cap U_j$. If $\gamma = \{U_i, f_i\}$ is an element of $H^0(X, \mathcal{K}^*/\mathcal{O}^*)$ or $H^0(X, \mathcal{K}^*/\mathcal{O}_P^*)$ we say γ *avoids* $\mathcal{P}$ if for all P_l, all the relevant f_i are units at P_l.

Let $H_P^0(X, \mathcal{K}^*/\mathcal{O}_P^*)$, respectively $H_P^0(X, \mathcal{K}^*/\mathcal{O}^*)$ be the subgroup of those γ which avoid all the P_l. The induced map $\rho: H^0(X, \mathcal{K}^*/\mathcal{O}_P^*) \to H^0(X, \mathcal{K}^*/\mathcal{O}^*)$ is onto and by definition $H_P^0(X, \mathcal{K}^*/\mathcal{O}_P^*)$ is the inverse image of $H_P^0(X, \mathcal{K}^*/\mathcal{O}^*)$. We need to prove Proposition 1.5 but we begin with Proposition 1.4.

Proposition 1.4. *Let X be of finite type over a Noetherian ring A with an ample bundle $\mathcal{J}$. Fix an integer m and a finite set of points P_l on X.*

(a) *Suppose $X = \mathbb{P}_A^r$. Then there is a homogeneous $f \in A[x_0, \ldots, x_r]$ of degree prime to m and not* 0 *at any P_l.*

(b) *There is a positive integer r and a section s of $\mathcal{J}^r$ such that r is prime to m, $\mathcal{J}^r$ is very ample, and the support of s contains none of the P_l.*

(c) *In particular, if X is projective, there is an affine open $U \subset X$ containing all the P_l.*

Proof. We begin with (a). Let $Q_l \subset A[x_0, \ldots, x_r]$ be the homogeneous prime ideals associated to the P_l. Our argument will be the standard one, watching the degrees as we proceed. We induct on s, the cardinality of the set of P_l. If $s = 1$, we can take f of degree 1. Assume the result for $s - 1$. Choose f_i of degree d_i, prime to m, such that $f_i \notin Q_j$ for $j \neq i$, $j = 1, \ldots, s$. We can assume $f_i \in Q_i$. Form $y = f_2^{t_2} \cdots f_s^{t_s}$ such that $d = d_2t_2 + \cdots + d_st_s$ is prime to m and all $t_i > 0$. Note that $y \in Q_i$ for $i > 1$ and $y \notin Q_1$. Consider $f = f_1^d + y^{d_1}$, which has degree d_1d prime to m. Then $f \notin Q_1$ because $y \notin Q_1$ and $f \notin Q_i$, $i > 2$, because $f_1 \notin Q_i$. Part (a) is done.

Next we claim there is an positive r, prime to m, such that $\mathcal{J}^r$ is very ample. This amounts to going through the proofs in [H], 7.6, which uses arguments of 5.14 and 5.4 in [H], and being slightly careful. But now part (b) reduces to (a). Part (c) is immediate. □

Proposition 1.5. *The maps $H_P^0(X, \mathcal{K}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*)$ and $H_P^0(X, \mathcal{K}^*/\mathcal{O}_P^*) \to H^1(X, \mathcal{O}_P^*)$ are surjective.*

Proof. Suppose $\mathcal{I}$ is a divisor on X, viewed as an element of $H^0(X, \mathcal{K}^*/\mathcal{O}^*)$. We have assumed X has an ample divisor $\mathcal{J}$. An easy argument from the definition (e.g., [H], p. 153) shows that $\mathcal{I} \otimes \mathcal{J}^n$ is ample for some n, and hence that $\mathcal{I}$ is the difference of ample divisors. Let $\mathcal{J}'$ be one of these ample divisors. By Proposition 1.4 there is a section of some $\mathcal{J}'^m$ whose support does not contain any of the P_l. Using Proposition 1.4 again, there is a section of $\mathcal{J}'^r$ whose support does not contain any of the P_l, where r is prime to m. Using a and b such that $ar + bm = 1$, it is clear that each $\mathcal{J}'$ is represented by a class in $H^0(X, \mathcal{K}^*/\mathcal{O}^*)$ which misses all the P_l, and so the same applies to $\mathcal{I}$.

That is, $H_P^0(X, \mathcal{K}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*)$ is surjective. Using the above diagram, it follows that $H_P^0(X, \mathcal{K}^*/\mathcal{O}_P^*) \to H^1(X, \mathcal{O}_P^*)$ is surjective. □

There is a well defined $\eta: H_P^0(X, \mathcal{K}^*/\mathcal{O}_P^*) \to \bigoplus_l k(P_l)^*$ given by evaluating the f_i at the relevant P_l. It is immediate that η is a splitting of the map $\rho: \bigoplus_l k(P_l)^* \to H_P^0(X, \mathcal{K}^*/\mathcal{O}_P^*)$ defined above. The inverse image of $H_P^0(X, \mathcal{K}^*/\mathcal{O}_P^*)$ in $H^0(X, \mathcal{K}^*)$ is K_P^*, defined as the subgroup of K^* of all functions which are units at all the P_l. The following is now clear:

Proposition 1.6. *Let $K_P^* \subset H_P^0(X, \mathcal{K}^*/\mathcal{O}^*) \oplus [\bigoplus_l k(P_l)^*]$ via $g \to ((X,g), \sum_l g(P_l))$. Then $H^1(X, \mathcal{O}_P^*)$ is the quotient*:

$$\frac{H_P^0(X, \mathcal{K}^*/\mathcal{O}^*) \oplus [\bigoplus_l k(P_l)^*]}{K_P^*}.$$

Note that if $\gamma \in H_P^0(X, \mathcal{K}^*/\mathcal{O}^*)$ has support within a locally factorial open subset of X, then γ can be identified with a (Weil) divisor whose support does not contain any of the P_l.

We can now use Theorem 1.2 to show the following. Let $S \to \mathrm{Spec}(\mathbb{Z}_p)$ be as usual with $\bar{S} \subset S$ the reduced closed fiber. Assume P_l are a finite set of closed points in $\bar{S}$ and m is an integer prime to p.

Proposition 1.7. *The canonical map induces an isomorphism*

$$\frac{H^1(S, \mathcal{O}_P^*)}{m(H^1(S, \mathcal{O}_P^*))} \cong \frac{H^1(\bar{S}, \mathcal{O}_P^*)}{m(H^1(\bar{S}, \mathcal{O}_P^*))}.$$

Proof. Given the exact sequence $0 \to \bigoplus_l k(P_l)^*/k^* \to H^1(X, \mathcal{O}_P^*) \to H^1(X, \mathcal{O}^*) \to 0$ above, to prove this isomorphism it suffices to prove that $H^0(S, \mathcal{O}^*) \to H^0(\bar{S}, \mathcal{O}^*)$ is onto. But by [H], p. 277, $H^0(S, \mathcal{O}) \cong \lim H^0(S_n, \mathcal{O})$ where S_n is the fiber of $p^n\mathbb{Z}_p$. Since units always lift modulo nilpotent ideals, we have the needed surjectivity. □

2. Classification of ramification

In this section we assume S is a nonsingular excellent surface. For any torsion abelian group A, $A' \subset A$ is the subgroup of elements of order prime to all residue characteristics of S. Let K be the field of fractions of S and $\alpha \in \mathrm{Br}(K)'$ an element of prime order q. We assume, for this section alone, that K contains a primitive q root of one ρ, which we fix. Using ρ, we define symbol classes etc. as in the introduction. For each curve $C \subset S$, the stalk $\mathcal{O}_{S,C}$ is a discrete valuation ring and so defines a ramification map $\mathrm{Br}(K)' \to H^1(F(C), \mathbb{Q}/\mathbb{Z})' = \mathrm{Hom}_c(G_{F(C)}, (\mathbb{Q}/\mathbb{Z})')$ where $F(C)$ is the residue field of $\mathcal{O}_{S,C}$ and $G_{F(C)}$ is the Galois group of $F(C)$ in its separable closure. As in the introduction, elements of $\mathrm{Hom}(G_{F(C)}, \mathbb{Q}/\mathbb{Z})$ are identified with pairs $L/F(C), \sigma$ where σ generates the Galois group of the cyclic extension $L/F(C)$. As observed above, the ramification locus of α is a finite union of curves on S. After blowing up (e.g., [L], p. 193), we can assume that this ramification locus consists of nonsingular curves with normal crossings.

What we study in this section includes the behavior of α with respect to *all* the discrete valuation rings R with $q(R) = K$ where R lies over points or curves on S. Note that if R lies over a curve of S, it equals $\mathcal{O}_{S,C}$ and so this is often not the hardest R to understand. Thus let P be a closed point of S, by which we mean a point of codimension 2. Let $R = \mathcal{O}_{S,P}$ be the stalk at P, which is a regular local ring of dimension 2. Let M/K be a cyclic Galois extension of degree q. We will be most interested in results about when M splits all the ramification of α over R.

We begin with a classification of the closed points of S with respect to their relationship to the ramification locus of α. Define $P \in S$ to be a *distant* point if it is not on the ramification locus of α. These points will rarely concern us. Define $P \in S$ to be a *curve* point if it is on a single irreducible curve of the ramification locus. Finally, define $P \in S$ to be a *nodal* point if it is a point in the intersection of two curves in the ramification locus. It is the nodal points that will mostly require our analysis. If $u \in R'$, and R' is a local ring, $\bar{u}$ is the image of u in the residue field $R'/\mathcal{M}'$ of R'. Let us quote a result from [S], p. 32, slightly reworded and in our special case.

Theorem 2.1. *Let α be as above, with ramification locus a union of nonsingular curves with normal crossings. If C is a curve in that locus, let $L_C/F(C), \sigma_C$ be the ramification data of α at C. Let $R = \mathcal{O}_{S,P}$ be the stalk at a curve or nodal point P. In the following, α' always refers to an element of* $\mathrm{Br}(R)$ *and u, v are always units in R.*

(a) *If P is a curve point and C is the curve in the ramification locus containing P, then in* $\mathrm{Br}(K)$, *$\alpha = \alpha' + (u, \pi)$ where $\pi \in R$ is a prime defining C at P.*
(b) *Suppose P is a nodal point contained in both C and C' among the ramification locus of α. Let π and δ be primes of R defining C, C' respectively at P. Then either* (i) *or* (ii) *below hold*:
   (i) $\alpha = \alpha' + (u, \pi) + (v, \delta)$.
   (ii) *There is an m prime to q such that $\alpha = \alpha' + (u\delta^m, v\pi)$.*

Furthermore, the following holds. In (a), *$L_C/F(C)$ is unramified at P and $\bar{u}^{1/p}$ defines $L_C/F(C), \sigma$ at that point. In* (b)(i), *$L_C/F(C)$ is unramified at P and also defined by $\bar{u}^{1/q}$ at P. In* (b)(ii), *$L_C/F(C)$ is ramified at P with ramification m/q and defined by $(u\delta^m)^{1/q}$ at P.*

In all cases above, we call $\alpha - \alpha'$ a *tail* of α at R or P.

We first consider the splitting at curve or distant points. The two cases are easy:

Theorem 2.2. *If P is a distant point, then α is unramified at any discrete valuation over P. Suppose P is a curve point on C, and C is in the ramification locus. Let $L/F(C), \sigma$ be the ramification data. If $L/F(C)$ splits at P, then α is unramified at any discrete valuation over P.*

Proof. The distant point case is obvious. Let P be a curve point on C. Write $\alpha = \alpha' + (u, \pi)$ where u is a unit at P with image $\bar{u} \in F(P)$. The residue field extension of $L/F(C)$ at P is defined by $F(P)(\bar{u}^{1/q})$. That is, $L/F(C)$ splits at P if and only if $\bar{u} \in (F(P)^*)^q$. Any valuation lying over P will have $F(P)$ as a subfield of its residue field, and so it is obvious that (u, π), and hence α, is unramified at all such. □

It will be considerably more complicated to understand splitting all ramification at a curve point P where $L/F(C)$ is not split. Let C be a curve along which α ramifies and P a nonsingular curve point on C. Let $R = \mathcal{O}_{S,P}$ and let $\pi = 0$ define C at P. Write $\alpha = \alpha' + (u, \pi)$ as above. Set $F(P)$ to be the residue field of R. Suppose $L/F(C), \sigma$ is the ramification data of α at C. Suppose $x = \pi^s \delta \in R$ with $(s, q) = 1$ and δ is prime to π in R. We are interested in when $M = K(x^{1/q})$ splits all the ramification of α over R. For convenience, may assume all the prime divisors of δ appear to prime to q powers. To state the next result we successively blow up to form $\rho : S' \to S$ in such a way as to resolve the singularities in the (reduced) support of $x = 0$ at P. Let $\{E_i\}$ be the exceptional fibers of ρ. Write $(x) = \sum_i r_i E_i + \sum_j s_j C_j + \sum_k t_k D_k$ where the C_j are strict transforms of curves in S containing P, and the D_k are the curves in S or S' not containing P. We may take $C = C_1$ and (by definition) $s = s_1$. We call a curve or point *relevant* if the residue field of that curve or point does not contain a q root of $\bar{u}$. Of course we call a point or a curve *irrelevant* if it is not relevant.

Theorem 2.3. *Let $\alpha = \alpha' + (u, \pi)$ be as above, and assume $L/F(C)$ is nonsplit at a curve point P. Further assume we have blown up to resolve the singularities of $x = 0$ as above, and so the r_i are defined. Then M does not split the ramification of α at P if and only if any of the r_i for relevant E_i are a multiple of q or, baring that, any of the intersection points among the union of the E_i and C_j are relevant.*

Proof. $L/F(C)$ is defined by $k(\bar{u}^{1/q})$ at P, and $\bar{u}$ is not a q power in $F(P)$. Thus P itself is relevant. It follows that all the strict transforms C_j are relevant. Also, by assumption, all the s_j are prime to q. Suppose there is a relevant E_i with r_i a multiple of q. An irrelevant exceptional curve can only be created by blowing up an irrelevant point, and that is not P. That is, the first exceptional curve created is relevant. We can assume E_i is the first relevant curve created in the resolution process with r_i a q multiple. Then E_i arises from blowing up a relevant point on a relevant $E_{i'}$ with $r_{i'}$ prime to q. Thus at the end of the process E_i will intersect $E_{i'}$ transversely at a relevant point P' with r_i and $r_{i'}$ as described and P' being on no other curves in the support of (x). Let $R_i = \mathcal{O}_{S',E_i}$. Then M/K is unramified at R_i. Define $K_i/k(E_i)$ to be the residue field extension of M/K at E_i. That is, $K_i = k(E_i)(\bar{y}^{1/q})$ where $\bar{y}$ is the image of some y which differs from x by a q power z^q, and y is a unit at E_i. It follows that y has prime to q valuation at $E_{i'}$, and hence that $K_i/k(E_i)$ ramifies at P'. Let $L_i/k(E_i), \sigma_i$ be the ramification data of α at E_i. Let d_i be the discrete valuation corresponding to E_i. Since d_i lies over P, the fact that $\alpha = \alpha' + (u, \pi)$ implies that $L_i = k(E_i)(\bar{u}^{1/q})$. Since L_i is unramified at P', it cannot equal K_i and M does not split the ramification of α with respect to d_i.

Next assume all the relevant E_i have r_i prime to q and P' is a relevant intersection point. Then P' is an intersection point of (say) local equations $\delta = 0$ and $\delta' = 0$ in the support of (x). Both curves are relevant. If $R' = \mathcal{O}_{S',P'}$, then $x = w\delta^s\delta'^t$ where $w \in R^*$ and s, t are prime to q. By Lemma 0.11 there is a valuation d lying over P' such that if $d(\delta) = a$ and $d(\delta') = b$ then $as + bt = nq$ and q does not divide ab. Thus M/K is unramified with respect to d. Since P' lies over P, just as above the ramification of α at d is $\bar{u}^{1/q}$. However, M can be described as $K((w^{s'}\delta^b/\delta'^a)^{1/q})$ where ss' is congruent to b modulo q. By Lemma 0.11 it is clear that the residue field of M does not contain $\bar{u}^{1/q}$ and once again we have a valuation where M does not split the ramification of α.

Conversely, suppose all relevant E_i have r_i prime to q and there are no relevant intersection points. Let d be a valuation lying over the original P. Then d must lie over a point or curve of the exceptional fiber. If d lies over an irrelevant point or irrelevant curve, α is unramified at d. Thus we may assume d lies over a relevant curve and since M/K ramifies there, it follows that M splits the ramification of α at any such d. Since M/K is also ramified at $\mathcal{O}_{S,C}$, we are done. □

The main reason for stating and proving Theorem 2.3 was to show how complicated our analysis would have to be if we had to analyze extension fields $M = K(x^{1/q})$ that are as general as occur there. The following case is much simpler.

Corollary 2.4. *Suppose, in the situation of Theorem 2.3, $x = u\pi^s\delta^q$ in $R = \mathcal{O}_{S,P}$ where $u \in R^*$ and s is prime to q. Then $M = K(x^{1/q})$ splits all the ramification of α.*

Proof. Here no blowing up is required and the result follows. □

We next classify what can happen at a nodal point P. Again set $R = \mathcal{O}_{S,P}$. If $\mathcal{M} \subset R$ is the maximal ideal, and $u \in R^*$, we let $\bar{u}$ be the image of u in $F = R/\mathcal{M}$. If case (b)(ii) of

Theorem 2.1 above holds we call P a *cold* point. We need to further analyze case (b)(i). Suppose $\bar{u}$, $\bar{v}$ do NOT generate the same subgroup of $F^*/(F^*)^q$. Then we say P is a *hot* point. If $\bar{u}, \bar{v}$ do generate the same subgroup of $F^*/(F^*)^q$, and they are not q powers in F, we say P is a *chilly* point. If $1 \leqslant s \leqslant q-1$ is such that $\bar{u}^s\bar{v}^{-1} \in (F^*)^q$ we say that s is the *coefficient* of this chilly point with respect to π. Of course viewing the curves in the other order, if s' is the coefficient of P with respect to δ, then ss' is congruent to 1 modulo q. If both u, v map to q powers in F, we say P is a *cool* point.

The rest of this section will be a study of these four kinds of nodal points. We begin the first of them.

Theorem 2.5. *Suppose P is a hot point. Then the residual classes of α are not split by the ramification. In particular, α has index larger than q.*

Remark. The Jacob–Tignol example in [S] of an exponent q and degree q^2 division algebra has a hot point, and the argument below is really theirs.

Proof. Write $\alpha = \alpha' + (u, \pi) + (v, \delta)$ as in Theorem 2.1(i). Since α ramifies at both π and δ, u is not a q power modulo π and v is not a q power modulo δ. We can assume the image of u is not a q power in $F = R/\mathcal{M}$. Let R' be the localization of R at (δ) with residue field F' and $L = K(\delta^{1/q})$. Let $\beta_{R'}$ be the residual Brauer class with respect to L. It is clear that $\beta_{R'} = \tilde{\alpha}' + (\tilde{u}, \tilde{\pi})$ where the tilde refers to images in $\mathrm{Br}(F')$ and F'^*. Then $\tilde{\pi}$ defines a discrete valuation on F', and with respect to this $\beta_{R'}$ has ramification $\bar{u}^{1/q}$, where $\bar{u}$ is the image of $\tilde{u}$ in F. The assumption that P is a hot point implies that $F(\bar{v}^{1/q})$ does not split this ramification. But $\tilde{v}^{1/q}$ is the ramification of α at δ, and we are done by Proposition 0.5. $\square$

Since in this paper we are concerned with division algebras of degree q, we often assume there are no hot points. Our next observation is that we can blow up to eliminate any cool points.

Theorem 2.6. *Suppose $P \in S$ is a cool point. Then if we blow up S at P, the Brauer group element α does not ramify on the exceptional divisor, and so the cool point has been turned into two curve points.*

Proof. Let $R, \mathcal{M}$ be the local ring of S at P, a cool point. Then a tail of α can be chosen to look like $[(u,\pi)_q] + [(v,\delta)_q]$ where $\bar{u}, \bar{v}$ are q powers in $F = R/\mathcal{M}$. If R' is a discrete valuation lying over $\mathcal{M}$ with valuation d, then the residue of this tail has the form $\bar{u}^{d(\pi)}\bar{v}^{d(\delta)}$ and so is a q power in F, which is a subfield of the residue field of R'. That is, α is unramified at every discrete valuation over $\mathcal{M}$, implying it is unramified on the exceptional divisor. $\square$

For the rest of this section we will assume we have used Theorem 2.6 to eliminate any cool points and that there are no hot points. Note that this means the following. Let P be an intersection point of two curves C, C' along which α ramifies with covers $L/F(C)$ and $L'/F(C')$. Then either P is a ramified point with respect to both extensions or P is a nonsplit point with respect to both extensions. We are left with studying chilly and cold points. Let us begin with chilly points.

Proposition 2.7. *Suppose P is a chilly point, $R = \mathcal{O}_{S,P}$ and $\pi \in R$, $\delta \in R$ are the two primes defining the ramification locus of α at P. Let s be the coefficient with respect to π, and w a unit of R.*

(a) *$M = K((w\pi\delta^s)^{1/q})$ splits all the ramification of α at any prime lying over R.*
(b) *For any t not congruent to s modulo q, $M' = K((w\pi\delta^t)^{1/q})$ fails to split the ramification of α at some prime lying over R.*

Proof. Suppose M is as described in (a). Let $d : M \to \mathbb{Z}$ be a valuation lying over R. If d lies over any height one prime not π or δ, or if $d(\pi)$ and $d(\delta)$ are both q multiples, then clearly α is unramified at d. If d lies over π or δ, then M/K is ramified at d and so M splits the ramification of α at d. Thus we may assume d lies over the maximal ideal, $\mathcal{M}$, of R, $d(\pi) = a > 0$ and $d(\delta) = b > 0$, and one of the a, b is prime to q. Note this also means that $k = R/\mathcal{M}$ is a subfield of the residue field of d. The ramification of (u, π) at d is $\bar{u}^{a/q}$ and the ramification of (v, δ) is $\bar{v}^{b/q} = \bar{u}^{bs/q}$, so the ramification of α is $\bar{u}^{(a+bs)/q}$. If $a + bs$ is prime to q, then M/K is ramified at d and so splits the ramification of α. If $a + bs$ is a multiple of q, α is not ramified at d and we are done.

Continuing with (b), let M' be as defined and $k = R/\mathcal{M}$. By Lemma 0.11 there is a valuation d on $K(T)$ lying over R with the following properties. First of all, $d(T) = 1$, and $d(\pi) + td(\delta) = mq$. Secondly, the residue field of d is $k(\pi', \delta')$ where $\pi' = \pi/T^{d(\pi)}$, $\delta' = \delta/T^{d(\delta)}$. Note that $x = w\pi\delta^t/T^{mq} = \pi'\delta'^t$ has image $\bar{x}$ which is part of a transcendence base $\bar{x}, z$ of $k(\pi', \delta')$ over k. Since $\bar{u}^t\bar{v}$ is not a q power in k, and $M'(T)$ has residue field $k(x^{1/q}, z)$ with respect to the unique extension d'', of d, it follows that the ramification of α is not split at d'' in $M'(T)$, and hence not split by the restriction of d'' to M'. □

Besides the splitting question handled above, we will need some results about the residual Brauer class in case (a) above.

Theorem 2.8. *Suppose P is a chilly point at the intersection of C and C' in the ramification locus of α. Let C, C' be locally defined by $\pi = 0$ and $\delta = 0$ respectively and let s be the coefficient with respect to C. Let $M = K((w\pi\delta^s)^{1/q})$ as in Proposition* 2.7(a) *above. Suppose β_C and $\beta_{C'}$ are the residual Brauer classes of α with respect to M/K. Then β_C and $\beta_{C'}$ are both unramified at P and have equal images in* $\mathrm{Br}(F(P))$.

Proof. Let $ss' - 1$ be divisible by q, so s' is the coefficient with respect to C'. We can also write $M = K((w^{s'}\delta\pi^{s'})^{1/q})$. At $R = \mathcal{O}_{S,P}$ write $\alpha = \alpha' + (u, \pi) + (v, \delta)$ where $\alpha' \in \mathrm{Br}(R)$, $u, v \in R^*$, and u^s and v differ by q powers in $F(P)$. Denote by $L/F(C), \sigma$ and $L'/F(C'), \sigma'$ the ramification data of α at C and C'. Then $L/F(C)$ is defined by $\bar{u}^{1/q}$ at P and $L'/F(C')$ is defined by $\bar{v}^{1/q}$ at P. The image of α in $\mathrm{Br}(M)$ is the same as the image of $\alpha'' = \alpha' + (u, w^{-1}\delta^{-s}) + (v, \delta) = \alpha' + (u, w^{-1}) + (v/u^s, \delta)$. Since v/u^s is a q power at P, the image of α'' in $\mathrm{Br}(F(C))$ is unramified at P. Moreover the image of α'' in $\mathrm{Br}(F(P))$ is $\bar{\alpha}' + (\bar{u}, \bar{w}^{-1})$. Looking at $\beta_{C'}$, which means reversing π and δ, and therefore switching s and s' and u, v, we get the image $\bar{\alpha}' + (\bar{v}, \bar{w}^{-s'})$ which is the same. □

Ultimately, we are going to show α is cyclic by finding an element f where the support of (f) includes the full ramification locus of α and the coefficients of (f) are chosen so that (a) in 2.7 above applies and not (b). There is an inherent difficulty with this if there are "loops" of curves where incompatible coefficients are required to meet condition (a) above. To get around this, we consider the effect of blowing up on a chilly point.

Let $[(u, \pi)_q] + [(v, \delta)_q]$ be a tail of α at $R = \mathcal{O}_{S,P}$ with coefficient s with respect to π. The blowup defines a valuation with $d(\pi) = d(\delta) = 1$, and so the ramification of α at the blowup is

$\bar{u}\bar{v}$ which is the same as $\bar{u}^{s+1}$ modulo q powers. Thus if $s+1$ is a multiple of q, there is no ramification on the blowup and we have turned a chilly point into two curve points. In any other case, there are 2 nodal points to consider. Let R' be the local ring at the intersection of the strict transform $\pi = 0$ and the exceptional divisor. Then in R' we have a ζ with $\zeta\delta = \pi$ where $\zeta = 0$ defines the strict transform of $\pi = 0$ and $\delta = 0$ defines the exceptional divisor. Thus the tail of α at R' is $(u, \zeta) + (uv, \delta)$. It follows that R' is a chilly point with coefficient $s+1$ with respect to $\zeta = 0$. Similarly, let R'' be the intersection of the exceptional divisor with the strict transform of $\delta = 0$ and let s' be the coefficient of P with respect to δ. The same argument shows that if P'' is the intersection of the exceptional divisor with $\delta = 0$, the coefficient is $s'+1$ at that point.

Consider a graph whose vertices are the curves in the ramification locus, and the edges are the chilly points. Two vertices have an edge between them if they both contain that chilly point. For any edge, blowing up can have one of two effects. If the coefficient is $q-1$, blowing up removes the edge. Otherwise, blowing up adds a vertex between the two vertices and two edges connecting the new vertex with both of the old ones. A loop in the above graph we call a *chilly loop*. It is clear that by repeated blowing up we can break any chilly loop.

Corollary 2.9. *After repeated blowing up, we can assume there are no chilly loops in the ramification locus of α.*

Corollary 2.10. *Suppose C_i are all the curves in the ramification locus and we have blown up so that there are no chilly loops. Then we can choose, for each C_i, a nonzero $s_i \in \mathbb{Z}/q\mathbb{Z}$ such that the following holds. Suppose P is a chilly point on C_i and C_j with coefficient s with respect to C_i. Then $s = s_j/s_i$ in $\mathbb{Z}/q\mathbb{Z}$.*

Proof. The graph is a tree so this is an easy induction, one leaf at a time. □

It now behooves us to consider splitting at cold points. More specifically, suppose P is a cold point defined locally by the intersection of curves C and C' in the ramification locus of α. Let $R = \mathcal{O}_{S,P}$ and let π and δ be primes of R defining C respectively C' at P. Suppose s, t are prime to q. We are interested in when $M = K((w\pi^s\delta^t)^{1/q})$ splits all the ramification of α over R. What we will find is that this is determined by the residual Brauer class β_C of Proposition 0.5. Recall that $\beta_C \in \mathrm{Br}(F(C))$, where $F(C)$ is the residue field of R localized at C. By assumption, P is nonsingular on C so defines a discrete valuation on $F(C)$. That is, if $F(P)$ is the residue field at P, β_C has some ramification $\chi_P \in \mathrm{Hom}(G_{F(P)}, \mathbb{Q}/\mathbb{Z})$.

Our immediate goal is a second description of χ_P in terms of ramification on $K = F(S)$. Let d' be a discrete valuation of K lying over P, and set $a = d'(\pi)$ and $b = d'(\delta)$. Let s' be the inverse of s modulo q. Assume M/K is unramified at d', which is equivalent to assuming $sa + tb$ is divisible by q. Let d be any extension of d' in M. Note that $F(P)$ is a subfield of the residue field of M at d.

Proposition 2.11. *Suppose P is a cold point and $M = K((w\pi^s\delta^t)^{1/q})$, β_C, χ_P, d are as above. The ramification of α at d is the image of χ_P^b.*

Proof. By Proposition 2.1 we can write $\alpha = \alpha' + (u^m\delta^m, v\pi)$ for m prime to q. Then α has the same image in $\mathrm{Br}(M)$ as $\alpha'' = \alpha' + (u^m\delta^m, vw^{-s'}\delta^{-s't})$ which is manifestly unramified with respect to C and so has image β_C in the residue field. In addition, α' maps to an element of $\mathrm{Br}(F(C))$ unramified at P. Finally, the image, $\bar{\delta}$, of δ in $F(C)$ is the prime defining P, and the images, $\bar{u}$, $\bar{v}$, $\bar{w}$, of u, v, w are all units at P. All together, χ_P is defined by $x^{1/q}$ where x is the image of $(\bar{u}^{m(-s't)}/\bar{v}^m(\bar{w}^{-s'm}))$ which up to q powers is $(\bar{w}/(\bar{u}^t\bar{v}^s))^{s'm}$. On the other hand, the ramification of α with respect to d is the image of the ramification of α'' with respect to d' and this (by the formula) is $y^{1/q}$ where y is the image of $(w/(u^tv^s))^{bs'm}$. $\square$

We can use the above calculations to observe a relationship between the ramification of the residual classes at cold points.

Corollary 2.12. *Suppose P is a cold point at the intersection of C, C' in the ramification locus. Let $M = K((w\pi^s\delta^t)^{1/q})$ be as above. Then the ramification of $s\beta_C$ and $-t\beta_{C'}$ are equal at P.*

Proof. Of course we have fixed a q root of one ρ, and it is easy to see that our description of the tail of α implies that $L/F(C), \sigma$ has ramification $\rho^{m'}$ at P, where $mm' - 1$ is divisible by q. By the proof of Proposition 2.11, and using ρ again, the ramification of β_C is represented by $x^{1/q}$ where x is the image of $(w/u^tv^s)^{ms'}$ and where $ss' - 1$ is divisible by q. To reverse the roles of C and C' we can also write $\alpha = \alpha' + (v^{-m}\pi^{-m}, u\delta)$. If $L'/F(C')$ is the ramification of α at C', then $L'/F(C')$ has ramification $\rho^{-m'}$ at P. The same argument as in Proposition 2.11 shows that $\beta_{C'}$ has ramification $x'^{1/q}$ where x' is the image of $(w/u^tv^s)^{-mt'}$. $\square$

The ramification of β_C at a cold point determines the splitting of the ramification of α at that point:

Corollary 2.13. *Suppose P is a cold point defined locally as the intersection of C and C' in the ramification locus. Let $R = \mathcal{O}_{S,P}$ and $M = K((w\pi^s\delta^t)^{1/q})$, for some s, t prime to q. Let β_C be the residual Brauer class of α with respect to M. Then M splits all the ramification of α over R if and only if β_C is unramified at P.*

Proof. Let $d : K^* \to \mathbb{Z}$ be a valuation over R at which α ramifies. If d lies over a prime of height one, it must be π or δ and M is ramified at those primes. Thus we may assume d lies over the maximal ideal of R. Let $d(\pi) = a > 0$ and $d(\delta) = b > 0$. If both a, b are divisible by q, α does not ramify at d, so we assume one of a or b is prime to q. If $sa + tb$ is not divisible by q, then M/K ramifies at d. Thus we may assume M/K is unramified at d. If χ_p is trivial, Proposition 2.11 shows that M splits the ramification at any such d.

Conversely, by Proposition 0.12, there is a valuation d on $K(T)$ where $sd(\pi) + td(\delta)$ is divisible by q and the residue field of d is $F(P)(\pi', \delta')$ as described there. If M splits the ramification of α at the restriction of that d, then $M(T)$ must split the ramification of α at d. In the notation of Proposition 2.11 it follows that (w/u^tv^s) must map to a q power in $F(P)(\pi', \delta')$ from which the result is clear. $\square$

While on the subject of residual Brauer classes, for completeness we add:

Corollary 2.14. *Suppose P is a curve point on C and the ramification $L/F(C)$ splits at P. Suppose $M = K(\pi^{1/q})$ and π has C valuation prime to q. If β_C is the residual Brauer class with respect to C, then β_C is unramified at P.*

Proof. Let $R = \mathcal{O}_{S,P}$. We can write $\alpha = \alpha' + (u, \pi_C)$ where $\pi_C = 0$ defines C locally at P. Also, $\pi = v\pi_C^s\delta$ where $v \in R^*$, s is prime to q, and δ is not divisible by π_C. α has the same image in $\mathrm{Br}(M)$ as $\alpha'' = \alpha' + (u, v''\delta')$ where $v'' \in R^*$ and δ'' is not divisible by π_C. Since $L/F(C)$ is split at P, the image, $\bar{u}$, of u in $F(P)^*$ is a q power. Direct calculation shows that since β_C is the image of α'', β_C is unramified at P. □

3. Adding a root of unity

The purpose of this section is to detail how the results of section two can be extended to the case where $K = F(S)$ does not contain a primitive q root of one. To this end, we begin more generally.

Let R be a regular local ring of dimension 2 with residue characteristic p, maximal ideal $\mathcal{M}$ and fraction field K. Let m be an integer prime to p. Let μ_m be the group of m roots of one over K with generator $\rho \in \mu_m$. Let $f(x)$ be the monic minimal polynomial of ρ over K, so $f(x) \in R[x]$ and we can set $R' = R[x]/(f(x))$. Then R'/R is Galois with (abelian) group H. For any prime of R, the group H acts transitively on the primes of R' lying over R.

Assume $\pi, \delta \in \mathcal{M} - \mathcal{M}^2$ and $\mathcal{M} = (\pi, \delta)$. Let $H_{\mathcal{M}}$, H_π, and H_δ be the stabilizers of one (and hence all) of the prime ideals lying over $\mathcal{M}$, (π), or (δ) respectively. If $R/\mathcal{M}$ contains a primitive m root of one, $H_{\mathcal{M}} = 1$. By Lemma 0.13 R' is a UFD:

Lemma 3.1. *One can choose π_1 such that π_1 generates a prime over (π) and such that the stabilizer of π_1 as an element is H_π. A similar result (of course) holds for (δ).*

Proof. R'^{H_π} is a UFD by Lemma 0.13. Let π_1 generate a prime of R'^{H_π} lying over π. □

By Lemma 3.1, we can write $\pi = u\prod_i \pi_i$ and $\delta = v\prod_j \delta_j$ to be the prime decompositions of π and δ in R' (where $u, v \in R'^*$). It is immediate that all the (π_i) and (δ_j) are distinct and each set forms a single H orbit. By Theorem 2.1(b), we can assume the sets of elements $\{\pi_i\}$ and $\{\delta_j\}$ form a single H orbit. Hence by changing our choice of π, δ we can assume $\pi = \prod_i \pi_i$ and $\delta = \prod_j \delta_j$. This is merely a convenience.

Let $\{\mathcal{M}_k\}$ be the set of maximal ideals of R' and $\mathcal{J} = \bigcap_k \mathcal{M}_k$ the Jacobson radical. Since R'/R is etale, $\mathcal{J} = (\pi, \delta)R'$. Any $\mathcal{M}_k$ contains π and δ and hence at least one π_i and one δ_j. Since $\mathcal{J}R'_{\mathcal{M}_k} = \mathcal{M}_k R'_{\mathcal{M}_k}$ we have $(\pi, \delta)R'_{\mathcal{M}_k} = (\pi_i, \delta_j)R'_{\mathcal{M}_k} = \mathcal{M}_k R'_{\mathcal{M}_k}$. If π_i and $\pi_{i'}$ were in the same $\mathcal{M}_k$, then $\pi \in \mathcal{M}_k^2$ which would contradict the above. Thus each $\mathcal{M}_k$ contains a unique π_i and δ_j. However, multiple $\mathcal{M}_k$ can contain the same π_i and δ_j. Checking locally, it follows that (π_i, δ_j) is the intersection of a uniquely defined set of maximal ideals of R'.

Lemma 3.2. *Let R'/R, $\pi = \prod_i \pi_i$ and $\delta = \prod_j \delta_j$ be as above. For a fixed $\mathcal{M}_k$, there is a unique π_i and a unique δ_j in $\mathcal{M}_k$. In particular, $H_{\mathcal{M}} \subset H_\pi \cap H_\delta$. The ideal $(\pi_i, \delta_j)R'$ is either R' or is the intersection of maximal ideals of R' which form a single and unique $H_\pi \cap H_\delta$ orbit.*

Proof. The inclusion $H_{\mathcal{M}} \subset H_\pi \cap H_\delta$ is immediate from the uniqueness of π_i, δ_j in some $\mathcal{M}_k$. Looking locally, it is clear that (π_i, δ_j) is the intersection of the maximal ideals containing it. This set of maximal ideals is clearly closed under the action of $H_\pi \cap H_\delta$. Assume $(\pi_i, \delta_j) \neq R'$. If $R'' = R'^{H_\pi \cap H_\delta}$, then R''/R is Galois with group $\bar{H} = H/(H_\pi \cap H_\delta)$. Every maximal ideal of R'' has trivial stabilizer. If $h \in H$ is not in $H_\pi \cap H_\delta$, then $h(\pi_i) \neq \pi_i$ or $h(\delta_j) \neq \delta_j$. Thus $h(\pi_i, \delta_j)R' \neq (\pi_i, \delta_j)R'$. It follows from a counting argument that $(\pi_i, \delta_j)R''$ is contained in a unique maximal ideal of R'', and so $(\pi_i, \delta_j)R''$ is a maximal ideal of R''. The rest of the lemma is now immediate. □

Let S be a nonsingular excellent surface. Let $S' \to S$ be the Galois cover gotten by adjoining a primitive q root of one, and let $K' = F(S')$ be the function field of S'. We assume this extension is etale, meaning that q is prime to all the residue characteristics of S. Let H be the Galois group of S'/S, so H is cyclic of order m dividing $q - 1$. The fact that m is prime to q is behind much of this section. For any curve $C \subset S$, or point $P \in S$, let H_C or H_P be the stabilizer of one and hence any point or curve lying over P and C respectively.

We are interested in applying section two to S' as a way of classifying ramification on S. To this end, we rephrase Lemmas 3.1 and 3.2 above. Let P be a point on S which is the local scheme theoretic intersection of nonsingular curves C and C', said curves locally defined by $\pi = 0$ and $\delta = 0$. Then by Lemmas 3.1 and 3.2, the points of S' mapping to P are each locally scheme theoretically defined by a unique C_i and C'_j in S', where the C_i lie over C and the C'_j lie over C'. In addition, none of the C_i intersect each other in S', and similarly for the C'_j.

Fix an element $\alpha \in \mathrm{Br}(F(S))$ of order q. Then, as usual, α has a ramification locus which is a bunch of curves C_i and cyclic covers $L_i/F(C_i), \sigma_i$ where σ_i generates the Galois group of $L_i/F(C_i)$. Also as usual, we can blow up and assume the C_i are all nonsingular with normal crossings. It will be important for us to understand the relationships between this ramification data and the corresponding data over S'. To this end, let α' be the image of α in $\mathrm{Br}(F(S'))$ and $L'_j/F(C'_j), \sigma'_j$ the ramification data of α'.

Theorem 3.3. *The C'_j are precisely the preimages of the C_i in S'. If C'_j lies over C_i, then $L'_j = L_i \otimes_{F(C_i)} F(C'_j)$. σ'_j is the extension of σ_i trivial on $F(C'_j)$. Furthermore, for fixed i, none of the inverse images of C_i intersect.*

Proof. This is obvious from Lemma 0.3, noticing that S'/S is unramified everywhere. $\square$

We will parallel section two and classify the points of S with respect to the ramification data of α. The easy things are easy. If P is not on any C_i, we say P is a *distant* point. If P is on exactly one C_i, we say P is a *curve* point. It is obvious from Theorem 3.3 that P is a distant or curve point if and only if one and hence all of its preimages in S' have the same behavior with respect to α'. It is also obvious that α is unramified at any discrete valuation over a distant point. The following is obvious from Theorem 2.2.

Theorem 3.4. *Suppose P is a distant point, or a curve point where the ramification $L/F(C)$ is split. Then α is unramified at any discrete valuation over P.*

We also need to generalize (trivially) Corollary 2.4.

Proposition 3.5. *Suppose P is a curve point on C, $R = \mathcal{O}_{S,P}$, and $x = u\pi^s\delta^q$ where $u \in R^*$ and s is prime to q. Then $M = K(x^{1/q})$ splits all the ramification of α over P.*

If P is on exactly two of the C_i, we say P is a *nodal* point. Then by 3.2 all the preimages of P are nodal points. Suppose P is on C_1 and C_2 and $L_k/F(C_k), \sigma_k$ is the ramification of α at C_k for $k = 1, 2$. Assume P' is a point on S' mapping to P and C'_1, C'_2 are curves on S' which lie over C_1, C_2 and both contain P'. Then $L_k/F(C_k)$ is ramified at P if and only if $L'_k = L_k \otimes_{F(C_k)} F(C'_k)$ is ramified at P'. If $L_k/F(C_k)$ is unramified at P, then P splits in L_k if and only if P' splits in L_k. Finally, if P extends uniquely in L_k, then the residue field of $L'_k/F(C'_k)$ at P' is the extension of that of $L_k/F(C_k)$ at P.

Theorem 3.6. *Let P be a nodal point. Then every preimage of P in S' is a nodal point for α'. If one of the preimage points of P is hot, or chilly, or cool, or cold, then all have the identical behavior. Furthermore, if all the preimages of P are chilly, then they all have the same coefficient with respect to the corresponding preimage of C_1.*

Proof. The result for cold points is obvious. In all other cases, the definitions of Section 2 looked at elements $\bar{u}, \bar{v} \in F(P)^*$. It suffices to use the observation of Theorem 2.1 that $F(P)(\bar{u}^{1/q})$ and $F(P)(\bar{v}^{1/q})$ are the residue extensions at P of the respective ramification extensions $L_1/F(C_1)$ and $L_2/F(C_2)$. □

Definitions. Clearly, then, it makes sense to say a point P of S is *hot*, or *chilly*, or *cool*, or *cold* if one and hence all its preimages have the same property.

It will also be useful to rephrase the condition of being a chilly point with coefficient s with respect to C_1. Suppose $L_1/F(C_1), \sigma_1$ and $L_2/F(C_2), \sigma_2$ is the ramification data for α at a nodal point P.

Corollary 3.7.

(a) *Suppose P is a chilly point with coefficient s with respect to C_1. Then both L_i are unramified at P. If $\bar{L}_i/F(P)$ are the induced residue extensions, then both are fields unramified at P and are equal. If $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are the induced generators of the Galois groups, then $\bar{\sigma}_2^s = \bar{\sigma}_1$.*
(b) *Suppose P is a cold point. Then both $L_1/F(C_1), \sigma_1$ and $L_2/F(C_2), \sigma_2$ are ramified at P. If ρ is the ramification of $L_1/F(C_1), \sigma_1$ at P, then ρ^{-1} is the ramification of $L_2/F(C_2), \sigma_2$ at P.*

Proof. Both parts are proven by extending to K' and using the fact K'/K has degree prime to q. In (a), we have just rewritten the definition of chilly and coefficient. In (b), we note the same fact at cold points in K' and again translate. □

There are a further series of consequences of Theorem 3.6.

Proposition 3.8. *If we blow up a cool point, then this point is replaced in the ramification locus by two curve points. After repeated blowing up, we can assume there are no chilly loops. With this, if C_i form the ramification locus of α then for each C_i we can choose a nonzero $s_i \in \mathbb{Z}/q\mathbb{Z}$ such that the following holds. Suppose P is a chilly point which is the locally the intersection of C_i and C_j, and which has coefficient s with respect to C_i. Then $s = s_j/s_i$ in $\mathbb{Z}/q\mathbb{Z}$.*

Proof. The blow up of a point P on S pulls back to the successive blow up (any order) of the preimage points. This makes the rest of the proposition clear. Since there are no chilly loops, the last sentence is clear just as in Corollary 2.10. □

Next we turn to generalizing Proposition 2.7. That is, we consider the coefficient at a chilly point and the consequences for splitting.

Proposition 3.9. *Let P be a chilly point and $\pi = 0$ and $\delta = 0$ the local equations for the two curves through P along which α ramifies. Let s be the coefficient with respect to π.*

(a) *$L = K((\pi\delta^s)^{1/q})$ splits all the ramification of α at any prime lying over R.*
(b) *For any t not congruent to s modulo q, $L_t = K((\pi\delta^t)^{1/q})$ fails to split the ramification of α at some prime lying over R.*

Proof. Obviously we will extend scalars to $K' = K(\mu_q)$ and use Proposition 2.7 to prove this. Let $R = \mathcal{O}_{S,P}$ and R'/R the extension gotten by adjoining μ_q. Let $\pi = \prod_i \pi_i$ and $\delta = \prod_j \delta_j$ be the prime decompositions in R'. At each closed point of R' defined by (π_i, δ_j), L and L_t have the form $K((wu_i\delta_j^s)^{1/q})$ and $K((w\pi_i\delta_j^t)^{1/q})$ respectively for a unit w at that point. Thus Proposition 2.7 applies. In (a), we conclude that $L' = L \otimes_K K'$ splits all the ramification over all points over P. Since L'/L has degree prime to q, we are done in (a). In (b), we find the discrete valuation over some preimage point that fails to kill the ramification and restrict it to K. □

We need to make some remarks about how adding roots of one effects residual Brauer classes. Suppose $P \in S$ is a nodal point and the intersection of C and C' in the ramification locus of α. Let $L/F(C), \sigma$ and $L'/F(C'), \sigma'$ be the associated ramification data. Let P_c be a preimage on S' of P, which is locally the intersection of C_c and C'_c which are preimages of C, C' respectively. Set $R = \mathcal{O}_{S,P}$, R' the ring gotten by adjoining a primitive q root of one to R, and R'_c the localization of R' at P_c. Let $\pi = 0$, $\delta = 0$ define C, C' at R and similarly for π_c, δ_c, C_c, C'_c and R'_c. Suppose $M = K((w\pi^s\delta^t)^{1/q})$ where s, t are prime to q. Since M splits the ramification of α at C and C', we can define the residual Brauer classes $\beta_C \in \mathrm{Br}(F(C))$ and $\beta_{C'} \in \mathrm{Br}(F(C'))$ with respect to M/K.

We are interested in describing $M' = M \otimes_K K'$ in terms of P_c. Since π_c appears to the first power in the R' factorization of π, and similarly for δ_c, we can write $M' = K'((w_c\pi_c^s\delta_c^t)^{1/q})$. Thus there are well defined residual Brauer classes $\beta_{C_c} \in \mathrm{Br}(F(C_c))$, $\beta_{C'_c} \in \mathrm{Br}(F(C'_c))$ of $\alpha' = \alpha \otimes_K K'$ at C_c and C'_c with respect to M'. The following is clear.

Proposition 3.10.

(a) *Under the natural maps induced by $F(C) \subset F(C_c)$ and $F(C') \subset F(C'_c)$, β_C maps to β_{C_c} and $\beta_{C'}$ maps to $\beta_{C'_c}$.*
(b) *Suppose P is a chilly point. Then β_C and $\beta_{C'}$ are both unramified at P and have equal images in* $\mathrm{Br}(F(P))$.
(c) *Suppose P is a cold point. The ramification of $s\beta_C$ and $-t\beta_{C'}$ are equal at P. M splits all the ramification of α at P if and only if the ramification of β_C is trivial at P.*
(d) *If α has a hot point, then the residual classes of α are not split by the ramification. In particular, α has index greater than q.*

Just as above, we can trivially extend Corollary 2.14 as follows.

Proposition 3.11. *Suppose P is a curve point on C and the ramification $L/F(C)$ splits at P. Suppose $M = K(\pi^{1/q})$ and π has C valuation prime to q. If β_C is the residual Brauer class at C with respect to M, then β_C is unramified at P.*

Proof. This is obvious by functoriality, Theorem 2.8, Corollaries 2.12 and 2.13. □

4. Killing the residual class

In Proposition 0.6 we saw how one can modify the residual class by changing the ramified extension. Next we observe how we can do that for several curves at once. To this end, let S be an excellent nonsingular surface projective over some affine A. Set $K = F(S)$ and suppose $\alpha \in \mathrm{Br}(K)$ is of order q.

We need to be slightly more general about the ramification locus. Let B be a finite set of curves on S including the ramification locus. As usual, suppose we have blown up S so that B consists of smooth curves with normal crossings. Let $\{P_j\}$ be the set of nodal points on the ramification locus, and assume we have further blown up so that there are no chilly loops and no cool points. Set $\{L_i/F(C_i), \sigma_i\}$ to be the ramification data of α. Suppose that for each C_i in the genuine ramification locus we fix s_i, as in Proposition 3.8, such that s_i is prime to q and the following holds. If P_j is a chilly point which is locally the intersection of C_i and C_j, and s is the coefficient of α at P with respect to C_i, then $s = s_j/s_i$ in $\mathbb{Z}/q\mathbb{Z}$.

Let $\mathcal{P}$ be a finite set of closed points including all nodal points of B. If any curve of B contains only finitely many closed points, we can assume $\mathcal{P}$ contains them all.

Lemma 4.1. *Let $\mathcal{P}$ be as above. We can choose $\pi \in K$ such that the support of $E = (\pi) - \sum_i s_i C_i$ contains no components of B, only intersects B in nonsingular points, and contains no point of $\mathcal{P}$.*

Proof. Use weak approximation to choose π' with valuation s_i at C_i. Write $(\pi) = \sum_i s_i C_i + E$. We can assume $\mathcal{P}$ includes a point on every component of B. By Proposition 1.5 there is a $u \in K$ with $(u) = E' - E$ where the support of E' does not contain any element of $\mathcal{P}$. Now $\pi = u\pi'$ is as needed. □

Let s_i and π be as in Lemma 4.1. Set $M = K(\pi^{1/q})$. Let β_{C_i} be the residual Brauer classes at C_i with respect to M. In the rest of this section we assume all the residual Brauer classes of α at the C_i are split by the ramification. By Proposition 0.5 this happens if α has index q. Note that this assumption means $\beta_{C_i} = \Delta(L_i/F(C_i), \sigma_i, u_i)$ for some $u_i \in F(C_i)^*$.

First we consider the ramification of the β_{C_i} at non-nodal points.

Theorem 4.2. *Let π be as above, and C_i some curve in the ramification locus of α. Let $\beta_{C_i} = \Delta(L_i/F(C_i), \sigma_{C_i}, u_i)$ be as above. Let P be a non-nodal point on C_i.*

(a) *If P is not in the support of E, then β_{C_i} is unramified at P.*
(b) *Suppose P is in the support of E. Then $L_i/F(C_i)$ is unramified at P. If $L_i/F(C_i)$ is split at P, then β_{C_i} is again unramified at P.*
(c) *Suppose P is in the support of E and $L_i/F(C_i)$ is not split at P. Let $\gamma = \bar{L}_i/F(P), \bar{\sigma}_{C_i}$ be the induced extension of $F(P)$ viewed as an element of $H^1(F(P), \mathbb{Q}/\mathbb{Z})$. Then the ramification of β_{C_i} has the form $-m_i(C_i.E)_P\gamma$ where $(C_i.E)_P$ is the intersection multiplicity at P and m_i is the modulo q inverse of s_i.*

Proof. Let $R = \mathcal{O}_{S,P}$. By the usual trick, it suffices to prove this theorem after adjoining a primitive q root of one, ρ, which we fix. Let $\pi_i \in R$ be a prime of R defining C_i locally at P. We know that $\alpha = \alpha' + (u, \pi_i)$ for some $u \in R^*$ and $\alpha' \in \mathrm{Br}(R)$. Then if $\bar{u}$ is the image of u in $F(P)$, $\bar{L}_i/F(P), \bar{\sigma}_i$ is the same as $F(P)(\bar{u}^{1/q})/F(P), \bar{\sigma}_i$ where $\bar{\sigma}_i(\bar{u}^{1/q})/\bar{u}^{1/q} = \rho$.

We turn to proving (a). Perhaps up to q powers, $\pi = v\pi_i^{s_i}$ where $v \in R^*$. It follows that for some $u' \in R^*$, $\alpha = \alpha'' + (u', \pi)$ where $\alpha'' \in \mathrm{Br}(R)$. The elements α and α'' have the same image in $\mathrm{Br}(M)$ and we can use α'' to compute β_{C_i}. Since $\alpha'' \in \mathrm{Br}(R)$, β_{C_i} is unramified at P.

Next we prove (b). Set E_P to be the sum $\sum t_j E_j$ over all E_j in the support of E which intersect C_i at P. For each E_j in the support of E_P let $\delta_j \in R$ be a prime such that $\delta_j = 0$ defines E_j at P. Set $\delta = \prod \delta_j^{t_j}$, the product over the support of E_P. Then, up to q powers, $\pi = v\pi_i^{s_i}\delta$ where $v \in R^*$.

Let $s_i m_i - 1$ be divisible by q, so up to q powers π_i is $\pi^{m_i}(v\delta)^{-m_i}$. The element α can be rewritten as $\alpha' + (u, (v\delta)^{-m_i}) + (u^{m_i}, \pi)$. As before, α has the same image in $\mathrm{Br}(M)$ as $\alpha' + (u, v^{-t_i'}\delta^{-t_i'})$ and the image of α' is unramified at P. If $L_i/F(C_i)$ is split at P, then $\bar{u}$ is a q power in $F(P)^*$ and β_{C_i} again is unramified at P. This proves (b). Otherwise by Lemma 0.12, the ramification of β_{C_i} is defined by $(\bar{u}^{-m_i n})^{1/q}$ where n is the valuation of $\bar{\delta}$ at P, and hence is $-m_i(C_i.E)_P\chi$. $\square$

We fix $\mathcal{Q}$ to be a finite set of closed points on the ramification locus which are on only one C_i and where the relevant β_{C_i} are unramified. If P is a point on a C_i and a component of B not among the C_i, then by Lemma 4.1 and Theorem 4.2 the relevant β_{C_i} is unramified at P and we can assume P is in $\mathcal{Q}$. Furthermore, by Lemma 4.1 and Theorem 4.2 we can assume that any curve among the C_i with no nodal points at all contains a point of $\mathcal{Q}$.

Proposition 4.3. *Let $\mathcal{Q}$ be a finite set of closed points as above. Assume all the residual Brauer classes of α, the β_{C_i}, are split by the ramification, so $\beta_{C_i} = \Delta(L_i/F(C_i), \sigma_i, u_i)$. In particular, assume there are no hot points. Let then there are $v_i \in F(C_i)$ such that*:

- (i) *The v_i are units at all nodal points and all the $Q_l \in \mathcal{Q}$.*
- (ii) *$\Delta(L_i/F(C_i), \sigma_i, v_i) = \Delta(L_i/F(C_i), \sigma_i, u_i^{s_i})$.*
- (iii) *If P is a nodal point and at the intersection of C_i and $C_{i'}$, then v_i and $v_{i'}$ have equal images in $F(P)$.*

Proof. Let P_j be a nodal point on C_i. Let $\hat{F}_j$ be the completion of $F(C_i)$ at P_j and let $\hat{L}_j = L_i \otimes_{F(C_i)} \hat{F}_j$. Define N_i to be the norm map of $L_i/F(C_i)$ and $\hat{L}_j/\hat{F}_j$. If $\hat{L}_j$ is split or ramified at P_j, then norms have all possible valuations, so we can choose $w_j \in \hat{L}_j$ such that $N_i(w_j)/u_i$ is a unit. If $\hat{L}_j$ is a field and unramified at P, then P must be a chilly point and $\Delta(L_i/F(C_i), \sigma_i, u_i)$ is unramified at P (Proposition 3.10). Thus u_i must have valuation a multiple of q and we can choose w_j such that $N_i(w_j)/u_i$ is a unit. At a point of $\mathcal{Q}$ we have assumed $\Delta(L_i/F(C_i), \sigma_i, u_i)$ is unramified and so once again w_l exists with $N_i(w_l)/u_i$ a unit at that point. By weak approximation we can find $w \in L_i$ such that $N_i(w)/N_i(w_j)$ is a unit at all nodal points P_j and all points of $\mathcal{Q}$. Then u_i can be replaced by $u_i/N_i(w)$ and we can assume all the u_i are units at all the nodal points and all the points of $\mathcal{Q}$.

For clarity's sake, set $v_i' = u_i^{s_i}$. Let $v_i'(P)$ be the image of v_i' in the residue field $F(P)$ of a point P on C_i (when defined). Suppose P_j is a nodal chilly point at the intersection of C_i and $C_{i'}$ with coefficient s with respect to C_i. Let $\bar{L}_{ij}/F(P_j), \sigma_{ij}$ be the residue extension of $L_i/F(C_i), \sigma_i$ at P_j. This is well defined, a field, and equal to $\bar{L}_{i'j}/F(P_j), \sigma_{ij'}^s$, by the definition of s and chilly point.

Lemma 4.4. *If P_j is a chilly point, $v_i'(P_j)$ and $v_{i'}'(P_j)$ differ by a norm of $\bar{L}_{ij}/F(P) = \bar{L}_{i'j}/F(P)$. If P_j is a cold point, $v_i'(P_j)$ and $v_{i'}'(P_j)$ differ by a q power.*

Proof. If P_j is a chilly point, we know by Propositions 3.10 and 3.9 that

$$\Delta\big(\bar{L}_{ij}/F(P_j), \sigma_{ij}, v_i'(P_j)\big) = s_i \Delta\big(\bar{L}_{ij}/F(P_j), \sigma_{ij}, u_i(P_j)\big)$$

equals

$$\begin{aligned} s_i \Delta\big(\bar{L}_{i'j}/F(P_j), \sigma_{i'j}, u_{i'}(P_j)\big) &= s_{i'} s \Delta\big(\bar{L}_{ij}/F(P_j), \sigma_{i'j}, u_{i'}(P_j)\big) \\ &= \Delta\big(\bar{L}_{ij}/F(P_j), \sigma_{ij}, v_{i'}'(P_j)\big) \end{aligned}$$

because $\sigma_{ij'}^{s} = \sigma_{ij}$ and $v_{i'}' = u_{i'}^{s_{i'}}$. By, e.g., [LN], p. 45, we are done for chilly P_j.

If P_j is a cold point, we know by Proposition 3.10 that

$$s_i \Delta\big(L_i/F(C_i), \sigma_i, u_i\big) \quad \text{and} \quad s_{i'} \Delta\big(L_{i'}/F(C_{i'}), \sigma_{i'}, u_{i'}\big)$$

have inverse ramifications at P_j. Moreover, by Corollary 3.7, $L_i/F(C_i), \sigma_i$ and $L_{i'}/F(C_{i'}), \sigma_{i'}$ have inverse ramifications at P_j. It follows from Lemma 0.10 that $v_i'(P_j)$ and $v_{i'}'(P_j)$ differ by a q power. □

We are ready to finish the proof of Proposition 4.3, which is now easy. Weak approximation implies that we can modify the $v_i'(P)$ by norms or q powers independently at all the nodal points and a all points in $\mathcal{Q}$. Proposition 4.3 is immediate. □

The point of Proposition 4.3 was to have enough compatibility among the v_i to do the following.

Proposition 4.5. *Let v_i and Q_l be as in Proposition* 4.3*, and continue the same assumptions on the residual Brauer classes and the lack of hot points. Then there is an affine $U \subset S$ with affine ring R and a $v \in R$ such that the following holds.*

(a) *U contains all nodal points, contains $\mathcal{Q}$, and contains a closed point on all the curves in B.*
(b) *If R_i is the affine ring of $U \cap C_i$, then $v_i \in R_i$.*
(c) *The element v is a unit at all curves of B, at all nodal points of B and maps to v_i for all i.*

Proof. We can choose a set $\mathcal{P}$ of closed points so that the following is true. First, the points of $\mathcal{P}$ are not on any C_i, have a point on any component of B not intersecting a C_i, and include all nodal points of B not on any C_i. Thus among the points of $\mathcal{P}$, $\mathcal{Q}$, and the nodal points of the ramification locus are all nodal points of B and at least one point on any component of B.

By Proposition 1.4 there is an affine open $U' \subset S$ containing $\mathcal{P} \cup \mathcal{Q}$, and containing all the nodal points of the C_i. Let P_n' be the set of poles of the v_i on $U \cap C_i$. There is an f defined on U which is 0 on all the P_n' and nonzero at all P_m, all the nodal points, and all the Q_l. We set $U = U_f'$. This finishes (a) and (b).

Let $Q_i \subset R$ be prime ideals corresponding to the C_i and the P_m. If Q_i corresponds to a P_m, let v_i be arbitrary nonzero. Note that the Q_i corresponding to a P_m are maximal and relatively prime to any other of the Q_i. Translating into commutative algebra, we have a ring R, prime ideals Q_i with no inclusions among them, and elements $v_i \in R/Q_i$ such that the following holds. First, $Q_i + Q_{i'}$ is either R, or a finite intersection of maximal ideals M_j and each maximal ideal contains at most two Q_i. Second, whenever M_j contains $Q_i + Q_{i'}$, v_i and $v_{i'}$ have equal images in R/M_j. Just using these facts, (c) is proven by induction on the cardinality of the set of Q_i. Of course one Q_i is trivial. Suppose v' is chosen for $Q_1, \ldots, Q_{n-1}$. Set $J = \bigcap_{i=1}^{n-1} Q_i$ and $I = Q_n$. We claim $I + J$ is the intersection of maximal ideals M_j where M_j contains Q_n and one of the Q_i, $i < n$. But $I + J \subset \bigcap_j M_j$ is clear, and equality can be shown by checking it locally. But $R/(I + J)$ is the direct sum of the R/M_j and (c) follows from the exact sequence $0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I + J) \to 0$. □

Theorem 4.6. *Let α, C_i, Q_l, and s_i be as above. Assume all the residual Brauer classes at all the C_i are split by the ramification, and hence that there are no hot points. Then there is a choice of $\pi \in K$, such that π has valuation s_i at the C_i, $E = (\pi) - \sum_i s_i C_i$ does not contain any nodal points of B, or any point in $\mathcal{Q}$, or any components of B in its support, and with respect to $M = K(\pi^{1/q})$, all of the residual Brauer classes β_{C_i} are trivial. Furthermore, $(C_i.E)_P$ is a multiple of q for all points P on the C_i where $L_i/k(C_i)$ is nonsplit.*

Proof. We find π' as in Lemma 4.1 and v as in Proposition 4.5. Then by Corollary 0.7 $\pi = v\pi'$ has all the residue classes split. The last sentence follows from Theorem 4.2(c). □

Combining Theorem 4.6 with Propositions 3.9 and 3.10 we have:

Corollary 4.7. *If M is as in Theorem* 4.6*, then M splits all the ramification of α at chilly and cold points.*

5. The proof

We have gone as far as we can assuming S is a fairly general surface. In this section, we return to the situation of section one and assume $S \to \operatorname{Spec}(\mathbb{Z}_p)$ is projective, regular, excellent, of finite type, with relative dimension one. Let $\bar{S}$ be the reduced subscheme defined by the preimage of the closed point of $\operatorname{Spec}(\mathbb{Z}_p)$. We assume $\alpha \in \operatorname{Br}(K(S))$ has order q, and we let B be the union of the ramification locus of α and $\bar{S}$. We can blow up S so that B consists of nonsingular curves with normal crossings, and so that there are no cool points or chilly loops. If C is any curve on S, then the residue field of C will be written as $k(C)$ to emphasize that it is a curve over a finite field or Spec of a p adic number ring. For each C_i in the ramification locus of α, let $L_i/k(C_i), \sigma_{C_i}$ be the ramification data. Until Corollary 5.2 we assume that all the residual Brauer classes of α are split by the ramification, and hence that there are no hot points.

Let π be as in Theorem 4.6 and write (again) $(\pi) = \sum s_i C_i + E$. Let $\bar{E}$ be the divisor which is the sum, with coefficients 1, of all the curves in $\bar{S}$. Let $\gamma \in \operatorname{Pic}(S)$ be the line bundle equivalent to the divisor class $-E$, and $\bar{\gamma} \in \operatorname{Pic}(\bar{S})$ its image. Then E and $\bar{E}$ only intersect in smooth points of $\bar{S}$ and so we can represent $\bar{\gamma}$ as a divisor using the intersection of $-E$ and $\bar{E}$. In particular, $\bar{\gamma}$ has the form $\sum_j q n_j Q_j + \sum_l n_l Q'_l$ where by Theorem 4.6 the Q'_l are either not on the ramification locus of α or are at points where $L_i/k(C_i)$ splits. For each of the Q'_l choose a geometric curve $E'_l \subset S$ whose unique closed point is Q'_l (Lemma 1.1). Set $E' = -E - \sum_l n_l Q'_l$. In the notation of Proposition 1.6, let P represent the set of all nodal points on B. Consider the element $\gamma' \in H^1(S, \mathcal{O}_P^*)$ represented, as in Proposition 1.6, by the divisor E' and the element 1 at all points in P.

The image, $\bar{\gamma}'$, of γ' in $H^1(\bar{S}, \mathcal{O}_P^*)$ lies in $qH^1(\bar{S}, \mathcal{O}_P^*)$. It follows from Proposition 1.7 that γ lies in $qH^1(S, \mathcal{O}_P^*)$. That is, using Proposition 1.6, there is a divisor E'', elements $a_j \in k(P_j)^*$ for all P_j in P, and an $f \in K = F(S)$ such that f is a unit at all nodal points, $(f) = E' + qE''$ and $f(P_j) = a_j^q$ at all P_j.

Now we compute the divisor $(f\pi) = \sum_i s_i C_i + \sum n_j D_j$. We note that for any curve D_j, D_j intersects B in a smooth point, and if n_j is prime to q, D_j either does not intersect any C_i, or does so at a point where $L_i/F(C_i)$ splits.

Theorem 5.1. *Let K be a field finite over $\mathbb{Q}_p(t)$. Let $\alpha \in \mathrm{Br}(K)$ have index a prime $q \neq p$. Then α is represented by a cyclic algebra of degree q.*

Proof. As in [S], we know K is the function field of a regular excellent projective surface S projective over $\mathrm{Spec}(\mathbb{Z}_p)$. As we have said before, we can blow up so that B, the union of the ramification locus of α and $\bar{S}$, consists of regular curves with normal crossings. We can further blow up so that the ramification locus has no cool points or chilly loops. By the assumption on the index, there are no hot points and the residual classes are all split by the ramification. Find π as Theorem 4.6. Choose f as above, and write $M = K((f\pi)^{1/q})$. For each curve C_i in the ramification locus, let β_{C_i} be the residual Brauer class of α at C_i with respect to M/K. We claim $\alpha' = \alpha \otimes_K M$ is not ramified on any discrete valuation over S.

The choice of s_i insures that α' is not ramified on the primes over the C_i, the curves in the ramification locus of α. Since α itself is unramified at all other curves, we are reduced to considering discrete valuations over points of S. By Theorem 3.4 we can also ignore distant points and curve points $P \in C_i$ where the ramification $L_i/F(C_i)$ splits. If $M' = K(\pi^{1/q})$, then by Theorem 4.6 all the residual classes with respect to M'/K are trivial. Since $f(P_j) \in (k(P_j)^*)^q$, it follows from Corollary 0.7, Propositions 3.9 and 3.10 that α' is unramified at any discrete valuation over a nodal point. Finally suppose P is a curve point on C_i where the ramification is nonsplit. By our choice of $f\pi$, the only curves in the support of $(f\pi)$ that meet P have coefficients a multiple of q. That is, if $R = \mathcal{O}_{S,P}$, then $f\pi = u\pi_c^s\delta^q$ where $u \in R^*$, $\pi_C = 0$ defines C locally at P, and s is prime to q. By Proposition 3.5, M splits all the ramification. By Theorem 0.9, M splits α, and so by Proposition 0.1 α is represented by a cyclic algebra of degree q. $\square$

One might be interested in how to detect those α of index q. The answer is not complicated.

Corollary 5.2. *Suppose S is as in this section, $K = F(S)$, and $\alpha \in \mathrm{Br}(K)$ has order q in the Brauer group. Assume S has been blown up so that the ramification locus of α consists of non-singular curves with normal crossings. Then α has index q if and only if there are no hot points.*

Proof. Up until the statement of Theorem 5.1 we only assumed that all the residual Brauer classes were split by the ramification. We did not make this part of Theorem 5.1 only because it would be clumsy to state. So to prove Corollary 5.2, it suffices to show that without hot points, all the residual Brauer classes are split by the ramification. Consider C in the ramification locus, and let $M = K(\pi^{1/q})$ where the C defined valuation of π is prime to q. Set β_C to be the residual Brauer class with respect to M, and let $L/k(C), \sigma$ be the ramification of α at C. Since β_C must have order q (or 1), by Theorem 0.8 to show L splits β_C it suffices to show L splits the residues of β_C at all points P. This is automatic at any point where the prime defining P does not split in

L (e.g., use Lemma 0.3). Thus it suffices to show β_C is unramified at all points where $L/k(C)$ splits. But this is Proposition 3.11. $\square$

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